BEGINNING ALGEBRA – Roots and Radicals – (revised summer, 2003 – Olson)
Packet to Supplement the Current Textbook - Part 1 – Review of Square Roots & Irrationals
(This portion can be any time before Part 2 and should mostly be review material.)

a Finding Square Roots.

Recall:

When we raise a number to the *second power*, c^2, we say it is **squared**.

► The **square** of a number is the number times itself.
In symbols, $a^2 = a \cdot a$.

For example,
The square of 5 is 25 because $5^2 = 5 \cdot 5 = 25$.
The square of –5 is $(-5)^2 = (-5)(-5) = 25$.

When we want to *find what number was squared*, we are **finding a square root** (the inverse of squaring).

► The *inverse* of squaring is **finding a square root**.

For example,
One square root of 25 is 5 because $5^2 = 25$.
The other square root of 25 is –5 because $(-5)^2 = 25$, **too**.

► **Squaring** and **finding a square root** are *inverse* operations.

► Every positive number will have **two** real-number **square roots**, (one **positive** and one **negative**).
The number **0** (zero) has just **one square root**, **0** itself.

Example 1 Find the square roots of 121.
The square roots of 121 are **11 and –11** because $11^2 = 121$ and $(-11)^2 = 121$.

Example 2 Find the square root of 0.
The only square root of 0 is 0 (since 0 is not positive or negative, so those choices don't exist here).

Now Do Practice Exercises 1 – 5.

Practice Exercises

1. Squaring and finding a square root are _____________ operations.

Find all square roots for each of the following.

2. 9 **3.** 0 **4.** 100 **5.** 144

► **Radical Notation:** $\sqrt{a}$

The pieces of a radical expression for **square roots:** $\sqrt{a}$, are

the **radical sign** or **radical:** $\sqrt{}$, and the **radicand**, *a*. (NOTE: The **radicand** is the entire expression under the **radical**, even if it's composed of several terms or factors).

► **Radical Notation**:

If $a > 0$ then $\sqrt{a} = b$, $b > 0$ and if $b^2 = a$.

In words, if *a* is a positive number, $\sqrt{a}$ is the **positive square root** or **principal square root of *a*** and $\sqrt{a}$ is equal to the ***positive*** **number *b*** whose square is *a*. $\sqrt{a}$ is read "the square root of a".

When the radical $\left(\sqrt{\ }\right)$ sign is used, to ask for the **negative square root**, a negative sign must be written **in front** of the radical:

$$-\sqrt{a} = -b, \quad b > 0 \quad \text{and if} \quad b^2 = a.$$

Also note: $\sqrt{0} = 0$.

Example 3 Find each square root.

a. $\sqrt{49}$ b. $\sqrt{1}$

Solution: a. $\sqrt{49} = 7$ because $7^2 = 49$.

b. $\sqrt{1} = 1$ because $1^2 = 1$.

Example 4 Find: $\sqrt{\dfrac{4}{25}}$

Solution: $\sqrt{\dfrac{4}{25}} = \dfrac{2}{5}$ because $\left(\dfrac{2}{5}\right)^2 = \dfrac{2}{5}\cdot\dfrac{2}{5} = \sqrt{\dfrac{4}{25}}$

Now Do Practice Exercises 6 – 13.

Practice Exercises

6. The radicand of $\sqrt{\dfrac{81}{7}}$ is ________.

7. The symbol, $\sqrt{\ }$, is called a _______.

Find each square root.

8. $\sqrt{100}$ 9. $\sqrt{64}$ 10. $\sqrt{121}$ 11. $\sqrt{0}$ 12. $\sqrt{\dfrac{1}{4}}$ 13. $\sqrt{\dfrac{9}{16}}$

b Approximating Square Roots.

So far, we've looked at square roots of perfect squares. Numbers like ¼, 36, $\frac{4}{25}$, and 1 are called **perfect squares** because they're the square of a whole number (or a fraction) and their square root is **rational** (see part **C** below for the formal definition of **rational**). A square root such as $\sqrt{5}$ cannot be written as a whole number since 5 isn't a perfect square. Such numbers are called **irrational** (see part **C** below for the formal definition of **irrational**).

Although $\sqrt{5}$ can't be written as a whole number, it can be **approximated** between two nearest wholes by finding the perfect squares surrounding the radicand or it can be **approximated** to about 12 decimal places **using a calculator**. Since 5 is between 4 and 9, $\sqrt{5}$ is between $\sqrt{4}$ and $\sqrt{9}$, i.e., $\sqrt{5}$ is between 2 and 3. Using a calculator, $\sqrt{5} \approx 2.236067977$. **Caution:** these are approximations and are not the exact value of $\sqrt{5}$. The exact value for $\sqrt{5}$ cannot be written as a whole number or a decimal, as it doesn't repeat or terminate, and is most simply written **exactly** as its radical form: $\sqrt{5}$.

Example 6 a) Find two whole values to approximate $\sqrt{23}$ between.

b) Use a calculator to approximate the $\sqrt{23}$ to the nearest thousandth.

Solution: a) Since 23 is between 16 and 25, $\sqrt{23}$ is between $\sqrt{16}$ and $\sqrt{25}$, or $\sqrt{23}$ is between 4 and 5 (but closer to 5).

On a scientific calculator, the square root key is usually a 2nd function above the x^2 (square) key.
For a scientific calculator (based on a TI-30), press: ON, 4, 3, 2nd, x^2 (and maybe =).
For a graphing calculator (based on a TI-82/83), you press: ON, 2nd, x^2, 4, 3, Enter.
You should see 4.795831523 appear.

NOTE: Most calculators are accurate to at least 3 more places than they display. The three more places for this problem are shown here: $\sqrt{23} \approx 4.795831523127$.

Since the nearest thousandth means 3 places after the decimal (or 3 decimal places), we look at the place to the right of that, and notice its an 8, which is greater than 5. So round the thousandths place up and drop values right of that.

To the nearest thousandth, $\sqrt{23} \approx 4.796$

Caution: $\sqrt{23}$ **= 4.796** is not correct, since it is an approximation. Whenever you round from your calculator, you should use the approximately equals symbol, ≈. So $\sqrt{23} \approx$ **4.796** is the correct notation. The only exact way to write $\sqrt{23}$ is $\sqrt{23}$, (or by using rational exponents – these will be discussed in future courses).

Now Do Practice Exercises 14 – 17.

Practice Exercises

14. Find two whole values to approximate $\sqrt{29}$ between.
15. Approximate $\sqrt{29}$ to the nearest thousandth.

Determine if each of the following is true or false. State a reason in either case.

16. The number $\sqrt{5} = 2.23$.
17. The number $\sqrt{64} = 8$.

C More on Rational vs. Irrational Numbers:

A **rational number** *q* is any number that can be written as a *ratio* of **integers** (or *quotient* or fraction of **integers**), where the denominator cannot be 0.

In **set-builder notation**: $\left\{ q \,\middle|\, q = \frac{a}{b},\right.$ where both *a* and *b* are integer and $b \neq 0$ $\}$

(read as "the set of numbers *q* such that *q* is equal to *a* divided by *b*, where both *a* and *b* are integer and *b* is not equal to zero").

The **decimal** form of a rational number always either **terminates** (i.e., ¾ = 0.75 stops with 5) or **repeats** (i.e., $1/3 = .3333...$, repeating the 3 forever).

Note: any **rational number** can always be written as a **fraction composed of integers**.
You may recall, if it is a terminating decimal, you put all the digits after the decimal over the digit 1 followed by that many zeros.

Example: For 0.75, put 75 over 1 followed by two 0's or 100 (and reduce). $0.75 = \frac{75}{100} = \frac{3}{4}$.

If it's a repeating decimal, put exactly one full repeat of it's digits over that same number of 9's.

Example: For 0.090909... = $0.0\overline{909}$, put one full repeat (09) over two 9's (and reduce).

$$0.0\overline{909} = \frac{09}{99} = \frac{9}{99} = \frac{1}{11} \text{ once it is reduced.}$$

More examples of **rational numbers** written in fraction and decimal form:

$$\frac{1}{3} = .333\overline{3}, \quad 2 = \frac{2}{1}, \quad \frac{3}{4} = 0.75, \quad \sqrt{9} = 3 = \frac{3}{1}, \quad 0 = \frac{0}{1}$$

Caution: some **rational number** decimal repeats cannot be seen as easily and **cannot be seen on a calculator at all**. For example, the number $1/17 = 0.0588235294117\overline{605882352941176}$ has 16 digits in a single repeat, but because 1/17 is a fraction of **integers**, it obeys the definition and is a **rational number**.

Reminder: any **natural number**, **whole number**, or **integer** can be put over 1, so all of these are also in the set of **rational numbers**.

An **<u>ir</u>rational number** is any number that is *<u>not</u> rational*. It cannot be written as a *ratio of* **integers** and its *decimal* representation neither **terminates** nor *repeats*.

Some examples of **irrational numbers** written in *exact* and *approximately equal (≈) decimal* form:

$\sqrt{2} \approx 1.414213562....$, $\pi \approx 3.141592654....$, $\sqrt{10} \approx 3.16227766....$,

0.101001000100001.... (This last one cannot be written in an exact form.)

NOTE: To get an accurate decimal form of any irrational number, use a scientific or graphing calculator, as they are not easily found by hand.

Caution: numbers like $\sqrt{9}$, which is exactly equal to 3 = $3/1$, and $1/17$, whose decimal repeat is long, are both still **rational numbers**, since they obey that definition (they can be written as a *ratio* of **integers**).

NOTE: *Most* **square roots** are **irrational**. Only **square roots** of *perfect squares* are **rational**.

The easiest way to tell if a number is **irrational** is to check that it is *not* **rational**. Ask your self these questions: Can it be written as some *ratio* (fraction) of **integers**? Or, if it's hard to tell that, check (on a calculator), does the decimal representation clearly **terminate** or **repeat**? (Again, these may not work if the repeat is a long one.)

Now Do Practice Exercises 18 – 20.

Practice Exercises

Determine if the following numbers are rational or irrational. Show your reasoning in either case.

18. $\frac{81}{7}$ 19. $\sqrt{43}$ 20. $\sqrt{9}$

Exercises for the Radicals Packet, Part 1:

a Find each square root.

1. 144 2. $\sqrt{121}$ 3. $-\sqrt{9}$

b Find two whole values to approximate each square root between.

4. $\sqrt{15}$ 5. $\sqrt{45}$

c Use a calculator to approximate each square root. Round the square root to the nearest thousandth.

6. $\sqrt{7}$ 7. $\sqrt{1100}$

8. $\sqrt{\frac{4}{9}}$ 9. $\sqrt{7}$ 10. .124124124124...

BEGINNING ALGEBRA – Roots and Radicals – (revised summer, 2003 – Olson)
Packet to Supplement the Current Textbook –
PART 2 - *n*th Roots & Simplifying Radical Expressions
(This portion can be done any time after laws of exponents have been done. Time in class ≈ 2 hr.)

a Square Roots, Reviewed and Expanded.

Recall:

When we raise a number to the *second power*, c^2, we say it is **squared**.
When we want to *find what number was squared*, we are **finding a square root**.

▸ The reversal (or inverse) of squaring is **finding a square root**.

The square roots of negatives:

▸ The square roots of negative numbers are **not real numbers.**

For example,

$\sqrt{-4}$ is *not a real number* - because there's no real number whose square is –4;
i.e., there is *no* real number x such that $x^2 = -4$ (since the square of any real number multiplies an even number of negatives, and is therefore always going to be positive (or zero)).

Square roots of negatives are imaginary numbers and will be discussed in later courses. Caution: they are **not** undefined, as they do exist and are used in some very real applications, e.g., the field of electronics.)

▸ **Radical Notation expanded**:
The (**principal**) **square root** of a positive number a is the *positive* number b whose square is a.
In symbols, if both $a > 0$ and $b > 0$,

$$\sqrt{a} = b \text{ if } b^2 = a,$$
$$-\sqrt{a} = -b \text{ if } b^2 = a,$$
$$\sqrt{0} = 0,$$
and $\sqrt{-a}$ is **not** a **real** number.

Example 1 Evaluate each of the following.

a. $\sqrt{49}$ b. $\sqrt{\frac{4}{25}}$ c. $\sqrt{-81}$

Solution: a. $\sqrt{49} = 7$ because $7^2 = 49$.

Solution: b. $\sqrt{\frac{4}{25}} = \frac{2}{5}$ because $\left(\frac{2}{5}\right)^2 = \frac{2}{5}\cdot\frac{2}{5} = \sqrt{\frac{4}{25}}$.

c. $\sqrt{-81}$ is not real because there is no real number whose square is –81.

Now Do Practice Exercises 1 – 4.

Practice Exercises

Evaluate.

1. $\sqrt{-100}$ 2. $\sqrt{0}$ 3. $\sqrt{\frac{1}{4}}$ 4. $\sqrt{.01}$ (Hint, recall .01 is $\frac{1}{100}$)

b *n*th roots.

As noted earlier, finding the square root is the inverse of squaring a number. Here we'll extend that idea to work with other roots of numbers.

For example,

The *cube root* of a number is the number we must cube (raise to the third power) to get that number (The cube root of 8 is 2 since $2^3 = 8$, and we write $\sqrt[3]{8} = 2$.)

the *fourth root* of a number is the number we must raise to the fourth power to get that number, etc.

> ▸ The n^{th} root of *a*, written $\sqrt[n]{a}$, is the number *x* where $x^n = a$.

As before, each part of a radical expression has a special name.

The parts of a radical expression for **any root**, are

the **index, *n*,** (the "root" number)

$\sqrt[n]{a}$,

the **radical sign** or **radical**, $\sqrt{\ }$, and the **radicand**, ***a***. (Recall, the **radicand** is the entire expression under the **radical**, even if it's composed of several terms or factors).

For example,

The *cube root* of 64 is written $\sqrt[3]{64}$

(Index is 3)

and it represents the number that would be cubed (or raised to the third power) to get 64.

So, $\sqrt[3]{64} = 4$ because $4^3 = 64$.

The *fourth root* of 81 is written $\sqrt[4]{81}$

(Index is 4)

It represents the number we would raise to the fourth power to get 81.

So, $\sqrt[4]{81} = 3$ because $3^4 = 3{\cdot}3{\cdot}3{\cdot}3 = 81$.

NOTE: When the **index** is **2**, it is a ***square root*** and it's not written: $\sqrt[2]{a} = \sqrt{a}$

> ▸ ***Odd*** **roots** of **negative numbers**, (where the **index** is ***odd***), will be both **real** and **negative**.

(This is true since an odd power multiplies an odd number of negatives, and is therefore always going to be negative. So a *negative real root* will exist in these cases.)

For example,

$\sqrt[3]{-64} = -4$ because $(-4)^3 = -64$.

> ▸ ***Even*** **roots** of roots of **negative numbers**, (where the **index** is ***even***), are still *not* real numbers.

For example, $\sqrt[4]{-16}$ is not real because there is no real number whose fourth power is –16;
i.e., there is *no* real number that can be raised to the fourth power and will result in a negative number.
This is because an even power of any real number multiplies an even number of negatives, and is therefore always going to be positive or zero.

The following table shows the most commonly used roots. You fill in the missing values.

Square Roots		**Cube Roots**	**Fourth Roots**	**Fifth Roots**
$\sqrt{1} = 1$	$\sqrt{\quad\quad} = 7$	$\sqrt[3]{1} = 1$	$\sqrt[4]{1} = 1$	$\sqrt[5]{1} = 1$
$\sqrt{4} =$	$\sqrt{64} =$	$\sqrt[3]{\quad\quad} = 2$	$\sqrt[4]{16} = 2$	$\sqrt[5]{32} = 2$
$\sqrt{\quad\quad} = 3$	$\sqrt{\quad\quad} = 9$	$\sqrt[3]{27} =$	$\sqrt[4]{81} = 3$	$\sqrt[5]{243} = 3$
$\sqrt{\quad\quad} = 4$	$\sqrt{\quad\quad} = 10$	$\sqrt[3]{64} =$	$\sqrt[4]{\quad\quad} = 4$	$\sqrt[5]{1024} = 4$
$\sqrt{25} =$	$\sqrt{121} =$	$\sqrt[3]{\quad\quad} = 5$	$\sqrt[4]{625} =$	
$\sqrt{36} =$	$\sqrt{144} =$			

Example 2 Evaluate each of the following.

a. $\sqrt[4]{256}$ b. $\sqrt[5]{1024}$ c. $\sqrt[3]{-125}$ d. $\sqrt[4]{-256}$

Solution: a. $\sqrt[4]{246} = 4$ because $4^4 = 256$

Solution: b. $\sqrt[5]{1024} = 4$ because $4^5 = 1024$

Solution: c. $\sqrt[3]{-125} = -5$ because $(-5)^3 = -125$

Solution: d. $\sqrt[4]{-256}$ is **not real** because there is no real number whose fourth power is –256.

Now Do Practice Exercises 5 – 8.

Practice Exercises

Evaluate if possible.

5. $\sqrt[3]{27}$ 6. $\sqrt[4]{16}$ 7. $\sqrt[4]{-81}$ 8. $\sqrt[3]{-1}$

To conclude this portion, we develop a general result needed in later courses. Let's start by looking at two examples.

$$\sqrt{2^2} = \sqrt{4} = 2, \quad \text{and} \quad \sqrt{(-2)^2} = \sqrt{4} = 2 \quad \text{since } (-2)^2 = 4$$

Consider the value of $\sqrt{x^2}$ where x is positive or negative.

In $\sqrt{2^2}$ where $x = 2$, $\sqrt{2^2} = 2$.

In $\sqrt{(-2)^2}$ where $x = -2$, $\sqrt{(-2)^2} \neq -2$.

Here $\sqrt{(-2)^2} = -(-2) = 2$ (the opposite of –2).

Comparing the results above, we see that

▸ $\sqrt{x^2}$ is x if x is positive or zero, and $\sqrt{x^2}$ is $-x$ (the opposite of x) if x itself is negative,

(i.e., raising to an *even power*, and then *taking an even root* forces all things positive or zero).

From your earlier work with absolute values you may remember that

$|x| = x$ if x is positive or zero, and $|x| = -x$ (the opposite of x) if x itself is negative.

(i.e., *absolute value* forces all things positive or zero).

Since this same sign pattern works for any even power and root, we can summarize the discussion by writing

▸ $\sqrt[n]{x^n} = |x|$ for *any even* index, ***n***, and *any real* number, ***x***.

Example 3 Evaluate each of the following.

a. $\sqrt{5^2}$ b. $\sqrt{(-4)^2}$

Solution: a. $\sqrt{5^2} = 5$ because $\sqrt{5^2} = \sqrt{25} = 5$ or $\sqrt{5^2} = |5| = 5$

Solution: b. $\sqrt{(-4)^2} = 4$ because $\sqrt{(-4)^2} = \sqrt{16} = 4$ or $\sqrt{(-4)^2} = |-4| = 4$

Now Do Practice Exercises 9 – 10.

Practice Exercises

Evaluate.

9. $\sqrt{7^2}$ 10. $\sqrt{(-7)^2}$

NOTE: Roots with odd indices *do not* require the absolute value, since an odd power (and so an odd number of negative signs multiplied) is involved.
For example:

$\sqrt[3]{5^3} = \sqrt[3]{125} = 5,$ because $5^3 = 125$

and $\sqrt[3]{(-5)^3} = \sqrt[3]{-125} = -5$ because $(-5)^3 = -125$

so

▸ $\sqrt[n]{x^n} = x$ where the index, *n,* is odd.

Combining the two concepts in one statement, we can say

▸ $$\sqrt[n]{x^n} = \begin{cases} |x| & \textit{where the index } n \textit{ is even} \\ x & \textit{where the index } n \textit{ is odd} \end{cases}$$

Example 4 Evaluate each of the following.

a. $\sqrt{(-4)^2}$ b. $\sqrt[3]{5^3}$ c. $\sqrt[3]{(-5)^3}$ d. $\sqrt[4]{(-2)^4}$

Solution: a. $\sqrt{(-4)^2} = |-4| = 4$ because $\sqrt[n]{x^n} = |x|$ when the index, *n*, is even (it's the unwritten 2 ro a square root here)

Solution: b. $\sqrt[3]{5^3} = 5$ because $\sqrt[n]{x^n} = x$ when the index, *n*, is odd (*n* = 3 here)

Solution: c. $\sqrt[3]{(-5)^3} = -5$ because $\sqrt[n]{x^n} = x$ when the index, *n*, is odd (*n* = 3 here)

Solution: d. $\sqrt[4]{(-2)^4} = |-2| = 2$ because $\sqrt[n]{x^n} = |x|$ when the index, *n* , is even (*n* = 4 here)

Now Do Practice Exercises 11 – 13.

Practice Exercises

Evaluate.

11. $\sqrt[3]{7^3}$ 12. $\sqrt[4]{(-2)^4}$ 13. $\sqrt[5]{(-1)^5}$

C Simplifying Radical Expressions.

For most applications, we need to put answers with radical expressions in "*simplest form*". The word simplest here just means the following three conditions have been met (the actual result may not look any "simpler" than when you started).

A radical expression is in simplest form when
1. There are no *perfect-power factors* (greater than or equal to the index) *inside* a radical.
2. No *fraction* appears *inside* a radical.
3. No *radical* appears in the *denominator* of a fraction.

As we will be sticking to *square roots* in this course, this alters the first condition to read:

1. There are no *perfect-square factors* under a *square root*.

For example:

$\sqrt{9}$ is *not* simplest form, since 9 is a perfect-square factor, and

$\sqrt{8}$ is *not* simplest form, since $\sqrt{8} = \sqrt{4 \cdot 2}$, and 4 is a perfect-square factor.

But $\sqrt{7}$ is simplest form, since 7 has *no* perfect-square factor (other than 1).

We need to look at two properties to simplify radical expressions. The first of these will be used to simplify expressions with perfect square factors under square root. (These properties are directly related to the "product law" and "quotient law" for exponents you studied previously, as you will see in the next course.)
First, look at the following example.

$$\sqrt{25 \cdot 4} = \sqrt{100} = 10$$

$$\sqrt{25} \cdot \sqrt{4} = 5 \cdot 2 = 10$$

This shows that $\sqrt{25 \cdot 4} = \sqrt{25} \cdot \sqrt{4}$. Here is the general law for this fact.

Product Property of Radicals

For any non-negative real numbers a, b, and positive real number n,

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

In words, the n^{th} root of a product is the product of the n^{th} roots (and visa versa).

Caution: $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ By letting a = 25 and b = 4, we can see why.

This makes the left side: $\sqrt{a+b} = \sqrt{25+4} = \sqrt{29}$,

whereas the right side becomes: $\sqrt{a} + \sqrt{b} = \sqrt{25} + \sqrt{4} = 5+2=7$,

and clearly $\sqrt{29} \neq 7$.

The first few perfect squares are listed here as a reminder to help you recognize them as factors:

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
X^2	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

NOTE: $a\sqrt{b} = a \cdot \sqrt{b}$ **and** $a \cdot \sqrt{b} = a\sqrt{b}$ **(The operation is multiply between a number or variable and a radical.)**

Example 4 Simplify each of the following.

a. $\sqrt{9}$ b. $\sqrt{8}$ c. $\sqrt{7}$ d. $\sqrt{108}$

Solution: a. $\sqrt{9} = 3$ since 9 is a *perfect square factor* itself

Solution: b. $\sqrt{8}$ Our goal here is to factor radicands as perfect squares and non-squares.

$= \sqrt{4 \cdot 2}$ since 8 = 4·2

$= \sqrt{4} \cdot \sqrt{2}$ by the Product Property of Radicals, (note: 4 is a *perfect square factor*)

$= 2 \cdot \sqrt{2}$ since $\sqrt{4} = 2$ (Evaluate the square roots of perfect squares only.)

$= 2\sqrt{2}$

Solution: c. $\sqrt{7}$ is already simplest form, since 7 has *no* perfect square factors (other than 1)

Solution: d. $\sqrt{108}$

$= \sqrt{9 \cdot 12}$ since 108 is divisible by 9

$= \sqrt{9 \cdot 4 \cdot 3}$ and then 12 is further divisible by 4.

$= \sqrt{9} \cdot \sqrt{4} \cdot \sqrt{3}$ by the Product Property of Radicals, (here, 9 and 4 are both perfect square factors)

$= 3 \cdot 2 \cdot \sqrt{3}$ since $\sqrt{9} = 3$ and $\sqrt{4} = 2$.

$= 6\sqrt{3}$

NOTE: If you didn't notice that 9 divides 108, you can always break the radicand down to it's ***prime factors*** to simplify the radical. If you use this method, you want to remove factor pairs from under the radical (since they produce perfect square factors).

Alternate Solution: d. $\sqrt{108}$

$= \sqrt{2 \cdot 2 \cdot 3 \cdot 3 \cdot 3}$ since the prime factorization of 108 = 2 · 2 · 3 · 3 · 3

$= \sqrt{2 \cdot 2} \cdot \sqrt{3 \cdot 3} \cdot \sqrt{3}$ by the Product Property of Radicals

$= \sqrt{2^2} \cdot \sqrt{3^2} \cdot \sqrt{3}$ since $2 \cdot 2 = 2^2$ and $3 \cdot 3 = 3^2$

$= 2 \cdot 3 \cdot \sqrt{3}$ since $\sqrt[n]{x^n} = |x|$ for even n

$= 6\sqrt{3}$

Now Do Practice Exercises 14 – 17.
Simplify.

Practice Exercises

14. $\sqrt{50}$ 15. $\sqrt{45}$ 16. $\sqrt{28}$ 17. $\sqrt{162}$

This process also works for variable expressions.

root of x^n becomes more simple:

- For any **non-negative** real number x and any integer $n > 1$, $\sqrt[n]{x^n} = x$.

Example 5 Simplify each of the following. Assume that all variables represent non-negative real numbers.

a. $\sqrt{4x^6}$ b. $\sqrt{125b^3}$ c. $\sqrt{108a^5}$

Solution: a. $\sqrt{4x^6} = \sqrt{4 \cdot x^2 \cdot x^2 \cdot x^2}$ since $x^{m+n} = x^m \cdot x^n$ (product law of exponents)

$= \sqrt{4} \cdot \sqrt{x^2} \cdot \sqrt{x^2} \cdot \sqrt{x^2}$ by the Product Property of Radicals

$= 2 \cdot x \cdot x \cdot x$ since x is assumed to be non-negative, $\sqrt[n]{x^n} = x$ for any n.

$= 2x^3$

Solution: b. $\sqrt{125b^3} = \sqrt{25 \cdot 5 \cdot b^2 \cdot b}$ since $b^{m+n} = b^m \cdot b^n$ (product law of exponents)

$= \sqrt{25}\sqrt{5} \cdot \sqrt{b^2} \cdot \sqrt{b}$ by the Product Property of Radicals

$= 5 \cdot \sqrt{5} \cdot b \cdot \sqrt{b}$ since $\sqrt{25} = 5$ and b is non-negative, $\sqrt[n]{b^n} = b$ for any n.

$= 5 \cdot b \cdot \sqrt{5} \cdot \sqrt{b}$ by commuting.

$= 5b\sqrt{5b}$ by the Product Property of Radicals.

Solution: c. $\sqrt{108a^5}$ Write the radicand as a product of squares times non-squares.

$= \sqrt{9 \cdot 4 \cdot 3 \cdot a^2 \cdot a^2 \cdot a}$ since 108 = 9 · 4 · 3 by earlier work,
and $a^{m+n} = a^m \cdot a^n$ (product law of exponents used repeatedly)

$= \sqrt{9} \cdot \sqrt{4} \cdot \sqrt{3} \cdot \sqrt{a^2} \cdot \sqrt{a^2} \cdot \sqrt{a}$ by the Product Property of Radicals (split the product up)

$= 3 \cdot 2 \cdot \sqrt{3} \cdot a \cdot a \cdot \sqrt{a}$ since $\sqrt{9} = 3$, $\sqrt{4} = 2$,
and $\sqrt[n]{a^n} = a$ when a is non-negative
(simplify the squared parts)

$= 3 \cdot 2 \cdot a \cdot a \cdot \sqrt{3} \cdot \sqrt{a}$ by commuting (to put any radicals at the right end)

$= 6a^2\sqrt{3a}$ write the repeated factors as an exponent,
and put the 3 and a under one radical
(using the Product Property of Radicals in reverse)

Now Do Practice Exercises 18 – 20.

Practice Exercises

Simplify. Assume that all variables represent non-negative real numbers.

18. $\sqrt{9x^4}$ 19. $\sqrt{8c^3}$ 20. $\sqrt{75m^5}$

So far we've only dealt with the first condition for simplest form (no square factors inside a square root). Before working on problems involving the second and third conditions, we'll need to look at another property.

Look at the following two expressions:

$\sqrt{\frac{16}{100}} = \sqrt{\frac{4}{25}} = \frac{2}{5}$ (by earlier work) and $\frac{\sqrt{16}}{\sqrt{100}} = \frac{4}{10} = \frac{2}{5}$.

Thus, $\sqrt{\frac{16}{100}} = \frac{\sqrt{16}}{\sqrt{100}}$ which gives us the next general rule.

Quotient Property of Radicals

For any non-negative real number a, and positive real numbers b and n,

$$\sqrt[n]{\frac{a}{n}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

In words, the n^{th} root of a quotient is the quotient of the n^{th} roots (and visa versa).

This property will be used to simplify radicals involving fractions to meet the last two conditions.

A radical expression is in simplest form when

2. **No *fraction* may remain *inside* a radical.**
3. **No *radical* may remain in the *denominator* of a fraction.**

Example 6 Simplify. Assume that all variables represent non-negative real numbers.
(Remember, that just means that $\sqrt{x^2} = x$ here, instead of $|x|$.)

a. $\sqrt{\frac{16}{4}}$ b. $\sqrt{\frac{2}{49}}$ c. $\sqrt{\frac{12x^2}{25}}$

Solution: a. $\sqrt{\frac{16}{4}}$ We must do something, since we can't leave a fraction under a radical.

$= \frac{\sqrt{16}}{\sqrt{4}}$ by the Quotient Property of Radicals

We're not finished, since now we have a radical in the denominator.

$= \frac{4}{2}$ since $\sqrt{16} = 4$ and $\sqrt{4} = 2$

We're still not done, since this reduces.

$= 2$ since 4÷2 = 2

NOTE, since 16 is divisible by 4, we could have performed that step first, (using the order of operations rules).

Alternate Solution: $\sqrt{\frac{16}{4}}$

$= \sqrt{4}$ since 16÷4 = 4 and **both are inside** the radical.

$= 2$ since $\sqrt{4} = 2$

Solution: b. $\sqrt{\frac{2}{49}}$ We must do something, since we can't leave a fraction under the radical.

$= \frac{\sqrt{2}}{\sqrt{49}}$ by the Quotient Property of Radicals

We're not finished, since now there's a radical in a denominator.

$= \frac{\sqrt{2}}{7}$ since $\sqrt{49} = 7$

Solution: c. $\sqrt{\frac{12x^2}{25}} = \frac{\sqrt{12x^2}}{\sqrt{25}}$ by the Quotient Property of Radicals

$= \frac{\sqrt{4 \cdot 3 \cdot x^2}}{\sqrt{25}}$ since 12 = 4 · 3

$= \frac{\sqrt{4} \cdot \sqrt{3} \cdot \sqrt{x^2}}{\sqrt{25}}$ by the Product Property of Radicals

$= \frac{2 \cdot \sqrt{3} \cdot x}{5}$ since $\sqrt{4} = 2, \sqrt{25} = 5$ and x is non-negative, $\sqrt[n]{x^n} = x$ for any *n.*

$= \frac{2x\sqrt{3}}{5}$ by commuting

Now Do Practice Exercises 21 – 23.

Practice Exercises

Simplify. Assume that all variables represent non-negative real numbers.

21. $\sqrt{\frac{25}{16}}$ 22. $\sqrt{\frac{7}{9}}$ 23. $\sqrt{\frac{50x^2}{49}}$

You may have noticed the above examples all had perfect squares in the denominator. If the denominator is *not* a perfect square, we will need to apply both the Quotient Property of Radicals and the Product Property of Radicals in a process called "*rationalizing the denominator*". This involves multiplying by the denominator over itself (still as a radical) to force the denominator to be a perfect square factor under the radical. This technique rewrites the fraction as its equivalent with a rational number in the denominator (and thus satisfying condition 3 – leave no radical in the denominator.)

For Example:

To simplify the following expression, $\sqrt{\frac{1}{2}}$, first use the Quotient Property of Radicals to rewrite the fraction under the radical as a quotient of radicals: $\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}$.

This is *not* simplest form, since there is a radical in the denominator (see condition 3). To remove it, we need to multiply by a factor that will make the denominator a perfect square. Here, multiplying by $\sqrt{2}$ would do the trick, since $\sqrt{2} \cdot \sqrt{2} = \sqrt{2 \cdot 2} = \sqrt{4} = 2$. But, we can't just arbitrarily introduce a factor of $\sqrt{2}$. **We need to use this factor in a fraction over itself** (to multiply by the number 1, which is ok).

▸ In general, if $a \geq 0$, then $\sqrt{a} \cdot \sqrt{a} = a$ (This follows from the Product Property of Radicals and the fact that $\sqrt{x^2} = x$ when x is non-negative.)

So, to finish the example, $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$ in effect, multiplying by **1** in the form $\frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$ fraction multiplication

$= \frac{\sqrt{2}}{2}$ since $\sqrt{2} \cdot \sqrt{2} = 2$

This example now meets all three conditions for simplest form.

Example 7 Simplify. Assume that all variables represent non-negative real numbers.

a. $\sqrt{\frac{5}{3}}$ b. $\sqrt{\frac{2x}{11}}$

Solution: a. $\sqrt{\frac{5}{3}} = \frac{\sqrt{5}}{\sqrt{3}}$ by the Quotient Property of Radicals

$= \frac{\sqrt{5}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{5} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$ multiplying by **1** in the form $\frac{\sqrt{3}}{\sqrt{3}}$ and fraction multiplication

$= \frac{\sqrt{5 \cdot 3}}{3}$ by the Product Property of Radicals and $\sqrt{x^2} = x$ for non-negative x

$= \frac{\sqrt{15}}{3}$

Solution: b. $\sqrt{\frac{2x}{11}} = \frac{\sqrt{2x}}{\sqrt{11}}$ by the Quotient Property of Radicals

$= \frac{\sqrt{2x}}{\sqrt{11}} \cdot \frac{\sqrt{11}}{\sqrt{11}} = \frac{\sqrt{2x} \cdot \sqrt{11}}{\sqrt{11} \cdot \sqrt{11}}$ multiplying by **1** in the form $\frac{\sqrt{11}}{\sqrt{11}}$ and fraction multiplication

$= \frac{\sqrt{2x \cdot 11}}{11}$ by the Product Property of Radicals and $\sqrt{x^2} = x$ for non-negative x

$= \frac{\sqrt{22x}}{11}$

Now Do Practice Exercises 24 – 26.
Simplify. Assume that all variables represent non-negative real numbers.

Practice Exercises

24. $\sqrt{\frac{1}{3}}$ 25. $\sqrt{\frac{4}{3}}$ 26. $\sqrt{\frac{3y}{7}}$

Exercises for the Radicals Packet, Part 2

a **b** *and Part 1 packet:*

Evaluate if possible.

1. $\sqrt{400}$
2. $-\sqrt{100}$
3. $\sqrt{-100}$
4. $-\sqrt{\frac{1}{25}}$
5. $\sqrt{-\frac{1}{25}}$
6. $\sqrt[3]{27}$
7. $\sqrt[4]{81}$
8. $\sqrt[3]{-27}$
9. $\sqrt[4]{-81}$
10. $-\sqrt[3]{64}$
11. $-\sqrt[3]{-8}$
12. $\sqrt[4]{625}$
13. $\sqrt[3]{1000}$
14. $\sqrt[3]{-\frac{8}{27}}$

Which of the following roots are rational numbers and which are irrational numbers?

15. $\sqrt{19}$
16. $\sqrt{36}$
17. $\sqrt[3]{9}$
18. $\sqrt[3]{8}$
19. $\sqrt[4]{16}$
20. $\sqrt{\frac{4}{9}}$
21. $\sqrt{\frac{4}{7}}$
22. $\sqrt[3]{-27}$
23. $-\sqrt[4]{81}$

b

Evaluate each of the following expressions.

24. $\sqrt{5^2}$
25. $\sqrt{(-5)^2}$
26. $\sqrt[3]{4^3}$
27. $\sqrt[4]{(-3)^4}$
28. $\sqrt[4]{2^4}$
29. $\sqrt[3]{(-5)^3}$
30. $\sqrt[5]{3^5}$
31. $\sqrt[5]{(-2)^5}$

a and **b**:

Find the two expressions that are equivalent. (Hint: evaluate each expression if possible first

32. $\sqrt{-16}$, $-\sqrt{16}$, -4
33. $-\sqrt{25}$, -5, $\sqrt{-25}$
34. $\sqrt[3]{-125}$, $-\sqrt[3]{125}$, $|-5|$
35. $\sqrt[5]{-32}$, $-\sqrt[5]{32}$, $|-2|$
36. $\sqrt[4]{10{,}000}$, 100, $\sqrt[3]{1000}$
37. 10^2, $\sqrt{10{,}000}$, $\sqrt[3]{100{,}000}$

c: Use the Product Property of Radicals to simplify the following expressions. Assume that all variables represent non-negative real numbers.

38. $\sqrt{18}$
39. $\sqrt{80}$
40. $\sqrt{125}$
41. $\sqrt{5x^2}$
42. $\sqrt{98m^4}$
43. $\sqrt{54a^5}$
44. $\sqrt{a^2b^5}$

Use the Quotient Property of Radicals to simplify the following expressions.

45. $\sqrt{\frac{4}{25}}$
46. $\sqrt{\frac{3}{4}}$
47. $\sqrt{\frac{5}{9}}$
48. $\sqrt{\frac{64}{49}}$

C: Use the Quotient and Product Properties of Radicals to simplify the following expressions. Assume that all variables represent non-negative real numbers.

49. $\sqrt{\frac{8a^2}{25}}$

50. $\sqrt{\frac{1}{5}}$

51. $\sqrt{\frac{3}{2}}$

52. $\sqrt{\frac{2x}{7}}$

53. $\sqrt{\frac{2x^2}{3}}$

54. $\sqrt{\frac{8s^3}{7}}$

Combining Concepts. ✎ This symbol means you need to write the answer in complete sentences.

✎ 55. Explain why $\sqrt{x^2}$ does not equal x for all real numbers. Give an example to support your reasoning.

✎ 56. Why is the use of absolute value not required for the expression $\sqrt[n]{x^n}$ when *n* is odd?.

✎ 57. Does the n^{th} root of x^2 always exist? Why or why not?

✎ 58. Why will writing $\sqrt{50} = \sqrt{10 \cdot 5}$ not help in writing the expression in simplified form?

Evaluate each of the following expressions. Assume that all variables represent *any* real number.

59. $\sqrt[1999]{(2a+b)^{1999}}$

60. $\sqrt[414]{(a+b)^{414}}$

✎ Decide whether each of the following is already in simplest form. If not, explain what needs to be done.

61. $\sqrt{10mn}$

62. $\sqrt{18ab}$

63. $\frac{\sqrt{98x^2 y}}{7x}$

64. $\frac{\sqrt{6xy}}{3x}$

Answers to Practice Exercises, Part 1

Page 1:
1. inverse
3. 0
5. 12 and –12

Page 2:
7. radical sign or radical
9. 8
11. 0
13. ¾

Page 3:
15. ≈5.385
17. Because $8^2 = 8{\cdot}8 = 64$, and $\sqrt{64}$ asks just for the positive root, $\sqrt{64} = 8$ is a true statement.

Page 4:
19. Since 43 is not a perfect square, $\sqrt{43}$ must be irrational.
Alternately, $\sqrt{43} \approx 6.557438524302$, which is a non-terminating, non-repeating decimal, so $\sqrt{43}$ must be irrational.

Answers to Exercises for the Radicals Packet, Part 1

Page 4:
1. 12 and –12
3. –3
5. 6 and 7
7. ≈33.166
9. (≈2.646, a non-terminating, non-repeating decimal, or, 7 isn't a perfect square), irrational

Answers to Practice Exercises, Part 2

Page 5:
1. not real
3. ½

Page 7:
5. 3
7. not real

Page 8:
9. 7
11. 7
13. –1

Page 10:
15. $3\sqrt{5}$
17. $9\sqrt{2}$

Page 11:
19. $2c\sqrt{2c}$

Page 12:
21. $\frac{5}{4}$
23. $\frac{5x\sqrt{2}}{7}$

Page 14:
25. $\frac{2\sqrt{3}}{3}$

Answers to Exercises for the Radicals Packet, Part 2

Page 14:
1. 20
3. not real
5. not real
7. 3
9. not real
11. 2
13. 10

15. ≈ 4.358898944 or 19 is not a perfect square, irrational
17. ≈ 2.080083823 or 9 is not a perfect cube, irrational
19. = 2, rational
21. ≈ .755928946 or 4/7 is not a perfect square, irrational
23. = –3, rational

25. 5
27. 3
29. –5
31. –2

in # 33-37, these results are in the order presented:

33. –5, –5, not real ⇒the first two are equivalent
35. –2, –2, 2 ⇒the first two are equivalent
37. 100, 100, ≈46.41588834 ⇒the first two are equivalent

39. $4\sqrt{5}$
41. $x\sqrt{5}$
43. $3a^2\sqrt{6a}$

45. $\frac{2}{5}$
47. $\frac{\sqrt{5}}{3}$

Page 15:
49. $\frac{2a\sqrt{2}}{5}$
51. $\frac{\sqrt{6}}{2}$
53. $\frac{x\sqrt{6}}{3}$

55. ✎ Answers vary
57. ✎ Answers vary

59. $2a + b$
60. $|a + b|$

✎ 61. This is already in simplest form.

✎ 63. This is not simplest since there's still a perfect square factor ($49x^2$) under the radical. The simplest form is $\sqrt{y}$.