Chapter 2: Functions, Equations, and Inequalities

2.4 Analyzing Graphs of Quadratic Functions

I. Quadratic Functions

$f(x) = ax^2 + bx + c$ is the standard form of a quadratic function.

$f(x) = a(x - h)^2 + k$ is the "nice" form of a quadratic function.

II. Important Characteristics of the Graph a Quadratic Function

A. The graph of a quadratic function is called a **parabola**.

B. The key point on a parabola is the **vertex**, (h, k).

C. If the parabola opens up, the vertex is a **minimum**.
If the parabola opens down, the vertex is a **maximum**.

D. The graph of a parabola is symmetric around a vertical line which cuts through the vertex. This line is called the **axis of symmetry**. Its equation is x = h.

E. The domain of every parabola is $(-\infty, \infty)$.
If the parabola opens up, its range is $[k, \infty)$. If the parabola opens down, its range is $(-\infty, k]$.

F. If a parabola opens up, it is decreasing on the interval $(-\infty, h)$ and increasing on the interval (h, ∞).

If a parabola opens down, it is increasing on the interval $(-\infty, h)$ and decreasing on the interval (h, ∞).

G. To find the x-intercepts of a parabola, set the equation equal to zero and solve for x.
To find the y-intercept of a parabola, plug in zero for x and solve for y.

Example 1A Given $f(x) = 2x^2 + 6x + 8$, find the following. (modified # 11 pg. 226)

a. the vertex. Is it a maximum or a minimum?

$h = -\frac{6}{2(2)} = -\frac{6}{4} = -\frac{3}{2}$ $k = f\left(-\frac{3}{2}\right) = 2\left(-\frac{3}{2}\right)^2 + 6\left(-\frac{3}{2}\right) + 8 = \frac{9}{2} - \frac{18}{2} + 8 = \frac{7}{2}$

vertex: $\left(-\frac{3}{2}, \frac{7}{2}\right)$ Since a > 0, the graph opens up, so the vertex is a minimum.

b. the equation of the axis of symmetry

$x = -\frac{3}{2}$

c. the domain and range

D: $(-\infty, \infty)$; R: $\left[\frac{7}{2}, \infty\right)$

d. the intervals where the graph is increasing / decreasing.

decreasing: $\left(-\infty, -\frac{3}{2}\right)$; increasing: $\left(-\frac{3}{2}, \infty\right)$

e. the x and y-intercepts

$2x^2 + 6x + 8 = 0$

$$x = \frac{-6 \pm \sqrt{(6)^2 - (4 \cdot 2 \cdot 8)}}{2(2)} = \frac{-6 \pm \sqrt{-28}}{4} = \frac{-6 \pm \sqrt{28}\,i}{4} = \frac{-6 \pm 2\sqrt{7}\,i}{4} = -\frac{3}{2} \pm \frac{\sqrt{7}}{2}i$$

Since there are no real solutions, f(x) does not have any x-intercepts.

y-intercept: (0, 8)

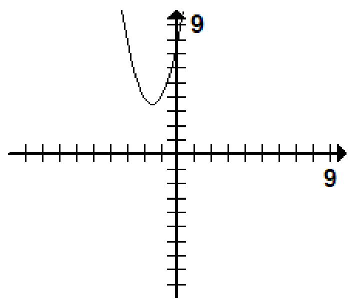
III. How to Graph a Quadratic Function

1. Find the vertex.
   a. To find the vertex of a parabola given in standard form, find $h = -\frac{b}{2a}$ and k = f(h).
   b. To find the vertex of a parabola given in "nice" form, identify h from the horizontal shift and k from the vertical shift.
2. Make the vertex the center of your t-chart. Plug in two x-values smaller than h and two x-values larger than h and find their corresponding y-values.
3. Plot all point and connect them with a smooth, continuous curve. Label one notch on each axis to indicate the scale you are using.

Example 1B Graph $f(x) = 2x^2 + 6x + 8$.

From Example 1A we know the vertex is: $\left(-\frac{3}{2}, \frac{7}{2}\right)$.

x	y
–3	8
–2	4
$-\frac{3}{2}$	$\frac{7}{2}$
–1	4
0	8



IV. How to Match the Graph of a Parabola to its Equation

1. Identify a, c, h, and k and the transformations they produce.
   a: x-axis reflection if a < 0;
   vertical stretch by a factor of a if |a| > 1; vertical shrink by a factor of a if 0 ≤ |a| ≤ 1.
   c: y axis reflection if c < 0;
   horizontal stretch by a factor of $\frac{1}{c}$ if 0 ≤ |c| ≤ 1; horizontal shrink by a factor of $\frac{1}{c}$ if |c| > 1.
   h: horizontal shift h units right if h > 0; h units left if h < 0
   x – h → h > 0 → right shift; x + h → h < 0 → left shift
   k: vertical shift k units up if k > 0; k units down if k < 0
2. Identify the h of the vertex from the horizontal shift and the k of the vertex from the vertical shift.
3. Look for the graph with the corresponding attributes.

Example 2 $y = -(x - 4)^2 + 3$

a = –1 → x-axis reflection; no vertical stretch or shrink
c = 1 → no y-axis reflection; no horizontal stretch or shrink
h = 4 → horizontal shift right 4
k = 3 → vertical shift up 3
vertex: (4, 3)
Graph e opens down and has its vertex at (4, 3)

V. Useful Formulas for Solving Quadratic Applications

Position equation: $s(t) = -16t^2 + v_0t + s_0$

s = height of object in feet. t = time in seconds. v_0 = the initial velocity or the velocity at time 0. s_0 is the initial height or the height at time 0.

Hint: The time at which the object reaches its maximum height is the x of the vertex.
The maximum height the object reaches is the y of the vertex.

Area of a Parallelogram: = $A = B \cdot H$ [Area = base · height]

Area of a Rectangle: $A = L \cdot W$

Perimeter of a Standard Rectangle: $P = 2L + 2W$

Volume of a Box: $V = L \cdot W \cdot H$

Cost Function: $C(x) = mx + b$, where m is the variable cost (cost per unit), x is the number of units, and b is the fixed costs.

Revenue Function: $R(x) = px$, where p is the price per unit and x is the number of units.

Profit Function: $P(x) = R(x) - C(x) = px - (mx + b) = px - mx - b$.
The **Break-Even Point** is when $R(x) = C(x)$ and $P(x) = 0$.

Example 3 A model rocket is launched with an initial velocity of 150 ft/sec from a height of 40 ft. The height of the rocket t seconds after it has been launched is given by the function $s(t) = -16t^2 + 150t + 40$. Determine the time at which the rocket reaches its maximum height and find the maximum height. (# 38 pg. 227)

t of vertex = $-\frac{150}{2(-16)} = \frac{75}{16}$ = 4.6875 seconds

s of vertex = $s\left(\frac{75}{16}\right) = -16\left(\frac{75}{16}\right)^2 + 150\left(\frac{75}{16}\right) + 40 = \frac{6265}{16}$ =391.5625 feet

The rocket reaches a maximum height of 391.5625 ft, 4.6875 sec after it is launched.

Example 4 The sum of the base and the height of a parallelogram is 69 cm. Find the dimensions for which the area is maximum. What is the maximum area? (# 42 pg. 227)

$B + H = 69 \rightarrow H = 69 - B \rightarrow A = B \cdot H = B(69 - B) = 69B - B^2$

$A(B) = -B^2 + 69B \rightarrow$ B of vertex = $-\frac{69}{2(-1)} = \frac{69}{2}$ = 34.5 cm H = 69 – 34.5 = 34.5 cm

A of vertex = $-(34.5)^2 + 69(34.5) = 1190.25$ cm^2

The maximum area of the parallelogram is 1190.25 cm^2. It occurs when the base and height are both 34.5 cm.

Example 5 Given $R(x) = 20x - .1x^2$ and $C(x) = 4x + 2$, find the maximum profit and the number of units that must be sold in order to yield the maximum profit. (# 46, pg. 228)

$P(x) = 20x - .1x^2 - (4x + 2) = 20x - .1x^2 - 4x - 2 = -.1x^2 + 16x - 2$.
$P(x) = -.1x^2 + 16x - 2 \rightarrow$ x of the vertex = $-\frac{16}{2(-.1)}$ = 80 units

P of the vertex = $-.1(80)^2 + 16(80) - 2$ = \$638

A maximum profit of \$638 is earned when 80 units are sold.