3.1 Polynomial Functions and Models

I. *Introduction to Polynomial Functions*

The equations of polynomial functions have "many terms" and the degree of the polynomial is determined by the highest exponent. For example, the

constant function	$f(x) = a$	has one term	and is	zero degree
linear function	$f(x) = ax + b$	has two terms	and is	first degree
quadratic function	$f(x) = ax^2 + bx + c$	has three terms	and is	second degree
cubic function	$f(x) = ax^3 + bx^2 + cx + d$	has four terms	and is	third degree
quartic function	$f(x) = ax^4 + bx^3 + cx^2 + dx + e$	has five terms	and is	fourth degree

We can write an n^{th} degree polynomial with n + 1 terms as

$P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + . . . + a_1x + a_0$.

a_n is called the <u>leading coefficient</u> and the term a_nx^n is called <u>the leading term</u>.

Hint: Be sure the terms of the polynomial are written in descending order of the exponents before you try to identify the leading term.

Example 1: Determine the leading term, the leading coefficient, and the degree of the given polynomial. Then classify the polynomial as linear, quadratic, cubic, or quartic. (#2 p. 264)

$f(x) = 15x^2 – 10 + .11x^4 – 7x^3$

After rearranging the terms of the function we have $f(x) = .11x^4 – 7x^3 + 15x^2 – 10$

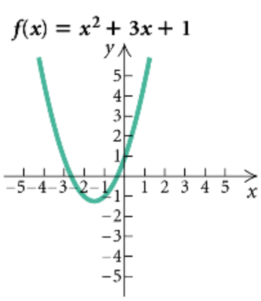
Leading term: $.11x^4$; leading coefficient: .11; degree: 4^{th}; quartic

II. *Characteristics of Polynomial Functions*

The graphs of all polynomial functions are smooth and continuous. They have no sharp corners and no holes or breaks.

The domain of every polynomial function is all real numbers, i.e. $(–\infty, \infty)$.

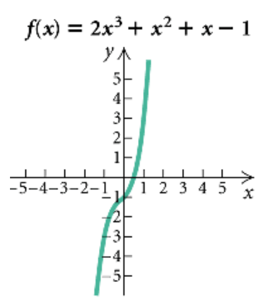
An n^{th} degree polynomial has at most *n* x-intercepts (real zeros) and at most *n – 1* turns (maxima and minima)



degree: 2

x-intercepts: 2

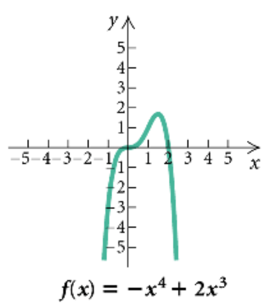
turns: 1



degree: 3

x-intercepts: 1

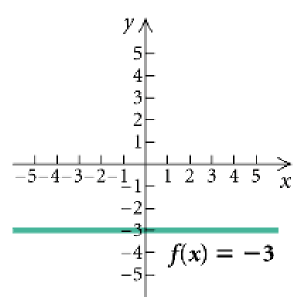
turns: 0



degree: 4

x-intercepts: 2

turns: 1



degree: 0

x-intercepts: 0

turns: 0

III. *The Leading Term Test*

Information about the "end behavior" of a polynomial function can be determined by looking at the leading term, $a_n x^n$. This is called the Leading Term Test. It is summarized in the table below.

	a_n is positive	**a_n is negative**
n is even	LH ↑ RH ↑	LH ↓ RH ↓
n is odd	LH ↓ RH ↑	LH ↑ RH ↓

The "hands" (ends of the graph) of even degree functions are together. If the leading coefficient is positive, they are both up. If the leading coefficient is negative, they are both down.

The "hands" of odd degree functions are opposed. If the leading coefficient is positive, the left hand is down and the right hand is up, i.e. the graph goes uphill from left to right. If the leading coefficient is negative, the left hand is up and the right hand is down, i.e. the graph goes downhill from left to right.

Example 2: Describe the end behavior of the graph of the given function. (#18 p. 265)

$f(x) = 2x + x^3 - x^5$

After rearranging the terms of the function we have: $f(x) = -x^5 + x^3 + 2x$

n is odd so the hands are opposed and a_n is negative so the graph is going downhill from left to right. Therefore LH ↑ RH ↓

IV. *Zeros, Intercepts, Solutions, and Factors*

A zero of function f is an input (x-value) which produces an output (y-value) of zero.

If c is a real zero of f, then (c, 0) is an x-intercept of the graph of f.

The zeros of a polynomial function f(x) are the solutions of the polynomial equation f(x) = 0.

If c is a zero of f, then x – c is a factor of f.

Example 3: Use substitution to determine whether 1, –2, & 3 are zeros of $g(x) = x^4 - x^3 - 3x^2 + 5x - 2$. (#26 p. 265)

$g(1) = (1)^4 - (1)^3 - 3(1)^2 + 5(1) - 2 = 0$ Yes, 1 is a zero of g(x) since g(1) = 0

$g(-2) = (-2)^4 - (-2)^3 - 3(-2)^2 + 5(-2) - 2 = 0$ Yes, –2 is a zero of g(x) since g(–2) = 0

$g(3) = (3)^4 - (3)^3 - 3(3)^2 + 5(3) - 2 = 40$ No, 3 is a not zero of g(x) since $g(3) \neq 0$

V. *Even and Odd Multiplicity of Zeros*

If a factor, (x – c), of a polynomial function occurs k times, we say the zero obtained from this factor has a multiplicity of k.

If k is odd, then the graph of the function crosses the x-axis at the point (c, 0).

If k is even, then the graph of the function is tangent to the x-axis at the point (c, 0), i.e. the graph touches, but does not cross, the x-axis at (c, 0).

Example 4: True or False?

If $P(x) = (x + 4)^2 (x - 1)^2$, then the graph of $y = P(x)$ is tangent to the x-axis at (4, 0). (#46 p. 266)

False. 4 is not a zero of P(x). The zeros of P(x) are 1 and –4. They both occur with a multiplicity of 2, so the graph is tangent to the x-axis at (1, 0) and at (–4, 0).

Example 5: Describe the end behavior of the graph. Find the zeros of the polynomial function and state the multiplicity of each. Also tell whether the graph crosses or is tangent to the x-axis at each zero. (#34 p. 266)

$f(x) = x^2 (x + 3)^2 (x - 4) (x + 1)^4$

$n = 9$ which is odd and $a_n = 1$ which is positive, so LH ↓ RH ↑

0 with a multiplicity of 2; the graph is tangent to the x-axis at x = 0

–3 with a multiplicity of 2; the graph is tangent to the x-axis at x = –3

4 with a multiplicity of 1; the graph crosses the x-axis at x = 4

–1 with a multiplicity of 4; the graph is tangent to the x-axis at x = –1

VI. *Finding the Zeros of a Factorable Polynomial Function*

1. Set the function equal to zero.
2. Factor the function completely.
3. Set each linear factor equal to zero and solve for x.
4. Record the value and multiplicity of each zero.

Example 6: Describe the end behavior of the graph. Find the zeros of the polynomial function and state the multiplicity of each. Also tell whether the graph crosses or is tangent to the x-axis at each zero.

$f(x) = -2x^3 - 5x^2 + 18x + 45$

Odd degree, negative coefficient ∴ LH ↑ RH ↓

$-2x^3 - 5x^2 + 18x + 45 = 0$

$-x^2(2x + 5) + 9(2x + 5) = 0$ Factoring by grouping.

$(2x + 5)(-x^2 + 9) = 0$

$2x + 5 = 0 \rightarrow x = -\frac{5}{2}$

$-x^2 + 9 = 0 \rightarrow -x^2 = -9 \rightarrow x^2 = 9 \rightarrow x = \pm\sqrt{9} = \pm 3$

$-\frac{5}{2}$, –3, & 3 all occur with a multiplicity of 1. Therefore, the graph crosses the x-axis at $-\frac{5}{2}$, –3, & 3.

VII. *Polynomial Models*

A. If given a polynomial model and an x-value, plug the given value in for every x in the function and solve for y.

B. If given a polynomial model and a y-value, plug the given value in for y and solve for x.

To do this graphically, enter the given polynomial function as y_1 and the given y-value as y_2. Then use 2nd Trace 5 (Intersect) to find the x-values of the points of intersection of y_1 and y_2.

Example 7:

The number of milk cows on farms in the United States peaked at 24,940,000 in 1940 and decreased to 9,005,000 in 2005. The quartic function given below can be used to estimate the number of milk cows, in thousands, from 1900 to 2005. x is the number of years since 1900. (#47 p. 266)

$f(x) = 0.0013924589x^4 – 0.2121392316x^3 + 3.414011377x^2 + 310.7243489x + 16{,}209.42591$

a. Find the number of cows in 1960.

$f(60) = 0.0013924589(60)^4 – 0.2121392316(60)^3 + 3.414011377(60)^2 + 310.7243489(60) + 16{,}209.42591 = 19{,}368$ → **19,368,000 cows**

b. In what year were there approximately 13,280,000 milk cows?

$y_1 = 0.0013924589x^4 – 0.2121392316x^3 + 3.414011377x^2 + 310.7243489x + 16{,}209.42591$

$y_2 = 13{,}280$

2nd Trace 5 (Intersect) $x = 75$ → **In 1975 there were approximately 13,280,000 milk cows.**

VIII. *Polynomial Functions*

A. To find a real zero (x-intercept) of a polynomial function, use 2nd Trace 2 (Zero or Root).

B. To find a relative minimum of a polynomial function, use 2nd Trace 3 (Minimum).

C. To find a relative maximum of a polynomial function, use 2nd Trace 4 (Maximum).

Example 8: Using a graphing calculator, find the real zeros and the relative maxima and minima of the given function. Then give the range of the function and the x-intervals on which it is increasing and decreasing. (#59 p. 268)

$f(x) = x^4 – 2x^2$

real zeros: –1.414, 0, 1.414

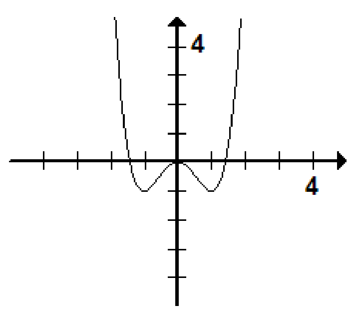
relative maxima: (0, 0)

relative minima: (–1, –1), (1, –1)

range: [–1, ∞)

decreasing: (–∞,–1), (0, 1)

increasing: (–1, 0), (1, ∞)



IX. ***Regression Analysis***

To fit a higher degree polynomial function to a set of data, follow the steps we used in section 1.4 for a linear regression analysis, but under STAT CALC select the appropriate polynomial function (quadratic, cubic, or quartic).

Example 9: The table below shows the number of unemployed (in thousands) in the United States from 1996 through 2006. x represents the number of years since 1996.

(#79 p. 269)

Year	x	Number of Unemployed
1996	0	397
1997	1	350
1998	2	340
1999	3	275
2000	4	269
2001	5	310
2002	6	360
2003	7	457
2004	8	460
2005	9	435
2006	10	411

a. Use a graphing calculator to fit cubic and quartic functions to the data. Let x represent the number of years since 1996.

$y_1 = -1.590909091x^3 + 27.21678322x^2 - 113.3951049x + 421.3426573$

$y_2 = -.2966200466x^4 + 4.341491841x^3 - 9.860722611x^2 - 39.24009324x + 399.986014$

b. Use the functions found in part (a) to estimate the number of unemployed in 2008. Compare the estimates and determine which model gives a more realistic estimate.

$y_1(13) =$ 230,730

$y_2(13) =$ –139,500

The cubic model gives a more realistic estimate.