# 3.4 Theorems about Zeros of Polynomial Functions

## *I. Key Concepts*

A. Definition of a Zero

A **zero** of function f is an input (x-value) which produces an output (y-value) of zero.

B. Multiplicity

A zero which is repeated m times is said to have a **multiplicity** of m.
If the multiplicity of a real zero is odd, the graph will cross the x-axis at that x-intercept.
If the multiplicity of a real zero is even, the graph will be tangent to the x-axis at that x-intercept.

C. Equivalent Statements

The following statements are equivalent:

- c is a *zero* of function f.
- x – c is a *factor* of f.
- c is a *solution* of the equation f(x) = 0.
- If c is a real number, (c, 0) is an *x-intercept* of the graph of f.
- f(c) = 0.
- 0 is the remainder when f(x) is divided by x – c.

## *II. Important Theorems*

A. The Fundamental Theorem of Algebra

Every polynomial function of degree n, with n ≥ 1, has at least one zero in the system of complex numbers.

B. Linear Factorization Theorem

Every polynomial function f of degree n, with n ≥ 1, can be factored into n linear factors (not necessarily unique); that is, $f(x) = a_n (x - c_1)(x - c_2) \dots (x - c_n)$.

C. In Other Words ...

In the complex number system, every nth-degree polynomial function has exactly n zeros and exactly n linear factors. Some of the zeros may be real, some may be complex, and some may be repeated.

## *III. Complex Zeros and Irrational Zeros*

A. Complex Zeros

For a polynomial function f with real coefficients, complex zeros occur in conjugate pairs. In other words, if a + bi is a zero of polynomial f, then a – bi is also a zero of polynomial f. The factors from these complex zeros would be x – (a + bi) = x – a – bi and x – (a – bi) = x – a + bi.

B. Irrational Zeros

For a polynomial function f with rational coefficients, irrational zeros also occur in conjugate pairs. If $a + c\sqrt{b}$ is a zero of polynomial f, then so is $a - c\sqrt{b}$. The factors from these irrational zeros would be $x - \left(a + c\sqrt{b}\right) = x - a - c\sqrt{b}$ and $x - \left(a - c\sqrt{b}\right) = x - a + c\sqrt{b}$.

**Example 1**

Suppose that a 4th degree polynomial with rational coefficients has the zeros (#24 p. 298)
$6 - 5i$ and $-1 + \sqrt{7}$. Find the other zeros.

The other two zeros are the conjugates of the given zeros, $6 + 5i$ and $-1 - \sqrt{7}$.

## IV. *Finding All the Zeros and All the Factors of a Cubic Function, Given One Zero*

1. Do synthetic division, using the given zero as the divisor.
2. Set the second degree quotient equal to zero and find its solutions.
3. Clearly list all zeros and their corresponding factors.

**Example 2** (like #43 p. 298)

Given –2 is one zero of the polynomial function $f(x) = x^3 + 2x^2 - 3x - 6$, find the other zeros.

$$\begin{array}{r|rrrr} -2 & 1 & 2 & -3 & -6 \\ & & -2 & 0 & 6 \\ \hline & 1 & 0 & -3 & \textcircled{0} \end{array}$$

Quotient: $x^2 - 3$

$x^2 - 3 = 0$

$x^2 = 3$

$x = \pm\sqrt{3}$

Other zeros: $-\sqrt{3}$, $\sqrt{3}$

## V. *The Rational Zeros Theorem (RZT)*

A. Rationale

If we are asked to find all the zeros of a non-factorable polynomial function, given no suggestions, we need a place to begin, a few zeros to try. The Rational Zeros Theorem helps us find all of the possible rational zeros of a polynomial function.

B. The Theorem

Assuming $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ is a polynomial function with integer coefficients, if we represent the factors of the constant, $a_0$, as *p* and the factors of the leading coefficient, $a_n$, as *q*, then all possible rational zeros of f will be in the form $\pm\frac{p}{q}$.

**Example 3**

List all the possible rational zeros of $f(x) = 10x^{25} + 3x^{17} - 35x + 6$. (#54 p. 298)

$a_0 = 6 \rightarrow p = \pm 1, 2, 3, 6$

$a_n = 10 \rightarrow q = \pm 1, 2, 5, 10$

All possible rational zeros of f(x) are:

$\pm 1, \frac{1}{2}, \frac{1}{5}, \frac{1}{10}, 2, \frac{2}{5}, 3, \frac{3}{2}, \frac{3}{5}, \frac{3}{10}, 6, \frac{6}{5}$

## *VI. Finding All the Factors and All the Zeros of a Polynomial Function Given No Zeros*

1. Try to factor. If factoring works, list all factors and zeros. If factoring doesn't work, do steps 2 – 6.
2. Use the RZT to list all the possible rational zeros.
3. Graph the polynomial on a graphing utility and use the graph to narrow the list of possible zeros to a list of probable zeros.
4. Test a probable zero via synthetic division. If the remainder is zero and the quotient is a cubic, do synthetic division a second time with a different possible zero on the cubic quotient.
5. When you have a zero remainder with a second degree quotient, set the quotient equal to zero and find the last two zeros by solving the quadratic equation.
6. Clearly list the actual zeros and their corresponding factors.

**Example 4**

For the polynomial function $f(x) = 2x^3 + 7x^2 + 2x - 8$, (#60 p. 299)

a. Find the rational zeros and then the other zeros; that is, solve f(x) = 0.

b. Factor f(x) into linear factors.

1) f does not factor by grouping

2) $a_0 = -8 \rightarrow p = \pm 1, 2, 4, 8$

$a_n = 2 \rightarrow q = \pm 1, 2$

All possible rational zeros are: $\pm 1, \frac{1}{2}, 2, 4, 8$.

3) From the graph we can see the probable zeros are: –2 and 1.

4)

| | | | | |
|---|---|---|---|---|
| -2 | 2 | 7 | 2 | -8 |
| | | -4 | -6 | 8 |
| | 2 | 3 | -4 | (0) |

Quotient: $2x^2 + 3x - 4$

$2x^2 + 3x - 4 = 0$

$$x = \frac{-3 \pm \sqrt{(3)^2 - (4 \cdot 2 \cdot -4)}}{2(2)} = \frac{-3}{4} \pm \frac{\sqrt{41}}{4}$$

5) rational zero: –2

other zeros: $\frac{-3}{4} + \frac{\sqrt{41}}{4}, \frac{-3}{4} - \frac{\sqrt{41}}{4}$

factors: $f(x) = (x + 2)\left(x + \frac{3}{4} - \frac{\sqrt{41}}{4}\right)\left(x + \frac{3}{4} + \frac{\sqrt{41}}{4}\right)$

**Example 5**

For the polynomial function $h(x) = 3x^4 - 4x^3 + x^2 + 6x - 2$, (#62 p. 299)

a. Find the rational zeros and then the other zeros; that is, solve $h(x) = 0$.

b. Factor $h(x)$ into linear factors.

1) f does not factor by grouping

2) $a_0 = -2 \rightarrow p = \pm 1, 2$

$a_n = 3 \rightarrow q = \pm 1, 3$

All possible rational zeros are: $\pm 1, \frac{1}{3}, 2, \frac{2}{3}$.

3) From the graph we can see the probable zeros are: $-1$ and $\frac{1}{3}$.

4)

$$\begin{array}{r|rrrrr} -1 & 3 & -4 & 1 & 6 & -2 \\ & & -3 & 7 & -8 & 2 \\ \hline & 3 & -7 & 8 & -2 & 0 \end{array} \qquad \begin{array}{r|rrrr} \frac{1}{3} & 3 & -7 & 8 & -2 \\ & & 1 & -2 & 2 \\ \hline & 3 & -6 & 6 & 0 \end{array}$$

$3x^2 - 6x + 6 = 0$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - (4 \cdot 3 \cdot 6)}}{2(3)} = \frac{6}{6} \pm \frac{\sqrt{-36}}{6} = 1 \pm 1i$$

5) rational zeros: $-1, \frac{1}{3}$

other zeros: $1 + 1i, 1 - 1i$

factors: $h(x) = 3(x + 1)\left(x - \frac{1}{3}\right)(x - 1 - 1i)(x - 1 + 1i)$

OR $h(x) = (x + 1)(3x - 1)(x - 1 - 1i)(x - 1 + 1i)$