4.2 Exponential Functions and Graphs

I. *Definition of an Exponential Function*

The <u>exponential function</u> f with base a is denoted by $f(x) = a^x$ where $a > 0$, $a \neq 1$, and x is any real number.

Examples: $f(x) = 3^{(1-x)}$ $\quad g(x) = \sqrt{2}^{x} - 4$ $\quad h(x) = \left(\frac{1}{5}\right)^{2x}$

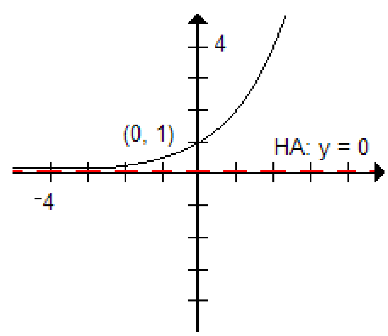
II. *Some Basic Information about Exponential Functions*

1. Exponential functions have a variable in the exponent, not the base. Thus, $f(x) = 2^x$ is an exponential function and $g(x) = x^2$ is not. g(x) is a quadratic function.
2. The base of an exponential function must be positive and cannot equal 1 since one to any power always equals one. $f(x) = 1^x$ would be a constant function, a horizontal line at y = 1.
3. $a^{-x} = \frac{1}{a^x} = \left(\frac{1}{a}\right)^x$ **Example**: $f(x) = 3^{-x} = \left(\frac{1}{3}\right)^x$
4. When evaluating an exponential expression on a calculator, we use the ^ key. If the exponent is not a single term, we must use parentheses around the exponent. If we'd like the answer in fraction form, we can use MATH 1, the FRAC feature, to return the answer as a fraction.

   Example: $4^{(-3+2)} - 1 \rightarrow$ 4^(−3 + 2) − 1 MATH 1 Enter $\rightarrow -\frac{3}{4}$

III. *Properties of Exponential Functions*

In general, a "plain vanilla" exponential function $f(x) = a^x$ has the following properties:



- "J" or banana shape
- continuous
- one-to-one
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- y-intercept at (0, 1)
- horizontal asymptote at y = 0 (the x-axis)
- increasing from left to right if $f(x) = a^x$ with $a > 1$ and decreasing from left to right if $f(x) = a^x$ with $0 < a < 1$ OR $f(x) = a^{-x}$ with $a > 1$

IV. *Transformations of Exponential Functions*

1. $f(x) = a^{x-h}$ horizontal shift right h units $\quad f(x) = a^{x+h}$ horizontal shift left h units
2. $f(x) = c\left(a^x\right), |c| > 0$ vertical stretch by a factor of c

   $f(x) = c\left(a^x\right), 0 < |c| < 1$ vertical shrink by a factor of c
3. $f(x) = -a^x$ x-axis reflection $\quad f(x) = a^{-x}$ y-axis reflection
4. $f(x) = a^x + k$ vertical shift up k $\quad f(x) = a^x - k$ vertical shift down k

   Note: A vertical shift will cause the horizontal asymptote to shift to y = k.

Example 1

Describe how $g(x) = 3 - 2^{-x+4}$ can be obtained from $f(x) = 2^x$.

c = –1 → x-axis reflection b = –1 → y-axis reflection

h = –4 → horizontal shift left 4 k = 3 → vertical shift up 3

V. Graphing Exponential Functions

1. Describe, the transformations which have occurred to the "plain vanilla" function.
2. Find and graph the horizontal asymptote. [y = k]
3. Find and graph the y-intercept. [Set x = 0 and solving for y.]
4. Set the exponent equal to zero and solve for x. This is the x of the "elbow" of the graph.
5. Make this x value the "center" of your t-chart. Choose two x's to the left and two x's to the right and find their y values.

   Hint: Remember to follow the rules of order - PEMDAS.
6. Plot these 5 points and connect them with a smooth curve using the HA as a guide.

Example 2

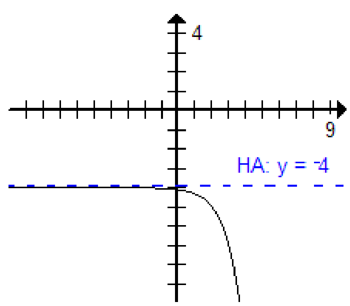
Graph $h(x) = -3^{x-2} - 4$

1. c = –1 → x-axis reflection b = 1 → no y-axis reflection
   h = 2 → horizontal shift right 2 k = –4 → vertical shift down 4
2. HA: y = –4
3. $y = -3^{(0-2)} - 4 \rightarrow y = -3^{(-2)} - 4 \rightarrow y = -\frac{1}{3^2} - 4 \rightarrow y = -\frac{1}{9} - 4 \rightarrow y = -\frac{37}{9}$

   y-intercept: $\left(0, -\frac{37}{9}\right)$
4. x – 2 = 0 → x = 2
5.

x	y
0	$-\frac{37}{9}$
1	$-\frac{13}{3}$
2	–5
3	–7
4	–13

6.



VI. The Compound Interest Formula $A = P\left(1 + \frac{r}{m}\right)^{mt}$

A = the future or accumulated balance
P = the present balance or principal
r = the interest rate (expressed as a decimal) Example: 7.5% = .075
m = the number of compounding per year
t = the number of years

annually: m = 1; semi-annually: m = 2; quarterly: m = 4; monthly: m = 12
weekly: m = 52; daily: m = 365 Note: Program CMPDINTR uses n instead of m.

Example 3

If the day Baby Beulah was born her parents invested $5000 for her at 6.5% compounded monthly, how much will Beulah have on her 18th birthday?

P = $5,000 r = .065 m = 12 t = 18

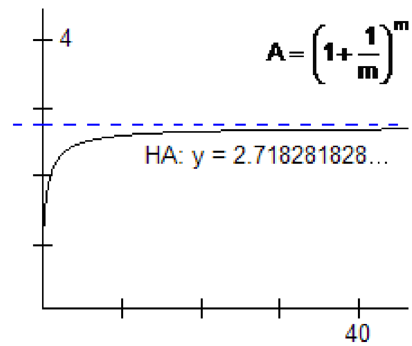
$$A = 5000\left(1+\frac{.065}{12}\right)^{(12\times 18)} = \$16{,}059.18$$

VII. The Natural Base, e, and the Natural Exponential Function, $f(x) = e^x$

Suppose $1 is invested for 1 year at 100% interest. The compound interest formula then simplifies to $A = \left(1+\frac{1}{m}\right)^m$. If we graph this function we can see that as the m values get larger and larger, the function gets closer and closer to 2.718281828.... This irrational number, which does not terminate or repeat is called the natural base or the natural number and it is denoted by the letter e in honor of Leonhard Euler. The function $f(x) = e^x$ is called the natural exponential function since it has a base of e.

m	A
1	2
2	2.25
4	2.44140625
12	2.61303529
52	2.692596954

m	A
365	2.714567482
8760	2.718126691
525,600	2.718279215
31,536,000	2.718282473
10,000,000,000	2.718281828



Note: On most graphers we press 2nd LN to access e^. Notice that the caret (^) is built into the function.

Example: $e^{\frac{1}{2}} \rightarrow$ 2nd LN (1 ÷ 2) Enter $\rightarrow$ 1.648721271

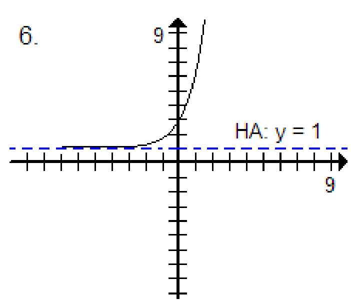
Example 4

Graph $f(x) = 2e^x + 1$.

1. c = 2 $\rightarrow$ vertical stretch by a factor of 2 b = 1 h = 0 k = 1 $\rightarrow$ vertical shift up 1
2. HA: y = 1
3. $y = 2e^0 + 1 = 2(1) + 1 = 3$ y-intercept: (0, 3)
4. x = 0
5.

x	y
–2	1.2707
–1	1.7358
0	3
1	6.4366
2	15.778

6.



VIII. The Continuous Compound Interest Formula $A = Pe^{rt}$

The variables have the same meaning as in the Compound Interest Formula. Notice there is no m since compounding is continuous. We will talk more about continuous compound interest in section 4.6.

Note: Program CONCMPD

IX. Other Applications of Exponential Functions

Example 5 (#57 p. 371)

$G(x) = 433.6(1.5)^x$ x = years since 2004 y = amount of data storage in gigabytes (GB)

2009 – 2004 = 5 $G(5) = 433.6(1.5)^5$ = 3292.65 $\approx$ 3293 GB

2014 – 2004 = 10 $G(10) = 433.6(1.5)^{10}$ = 25,003.56094 $\approx$ 25,004 GB