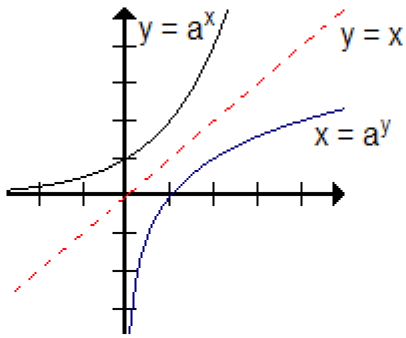
4.3 Logarithmic Functions and Their Graphs

I. Logarithmic Functions

A. Introduction

The graph of the exponential function $y = a^x$ looks like a J shape. It passes both a vertical and a horizontal line test which means it is a one-to-one function and it has an inverse.

To find the inverse of a function we can reflect it over the y = x line and interchange the x and the y. Thus the inverse of $y = a^x$ would be $x = a^y$ and its graph would look like an r shape.



Since we usually write functions with the y isolated on one side, we need to solve $x = a^y$ for y. To do that we will need a new symbol, something called a logarithm.

B. Definition of a Logarithmic Function

$y = \log_a x$ if and only if $x = a^y$

In words, $\log_a x$ is the exponent (y) we must raise the base (*a*) to in order to get the argument (x).

Example: $\log_2 8 = \square$ implies $2^{\square} = 8$.

To evaluate this logarithm, we ask the question "2 raised to what power equals 8?" Since $2^3 = 8$, $\log_2 8 = 3$.

Note: $y = \log_a x$ and $x = a^y$ are the same function in two different forms.

$y = \log_a x$ and $y = a^x$ are inverse functions which undo one another.

C. Comparing Exponential and Logarithmic Functions

Exponential Function	**Logarithmic Function**
$y = a^x$ with $a > 0$ & $a \neq 1$	$y = \log_a x$ with $a > 1$
continuous, 1-1, J shaped	continuous, 1-1, r shaped
Domain: $(-\infty, \infty)$ Range: $(0, \infty)$	Domain: $(0, \infty)$ Range: $(-\infty, \infty)$
Horizontal asymptote at y = 0 (x-axis)	Vertical asymptote at x = 0 (y-axis)
no x-intercept, y-intercept at (0, 1)	x-intercept at (1, 0), no y-intercept
increasing if a > 1, decreasing if 0 > a > 1	increasing if a > 1, decreasing if 0 > a > 1

Remember, when all is said and done, **a logarithm is an exponent**.

II. Common Logarithms and Natural Logarithms

A. Common Logarithms

Base ten logarithms are called **common logarithms**. They are denoted manually as $\log_{10}$ or log and as LOG on a calculator.

The inverse of the common log function, $y = \log_{10} x$, is the exponential function $y = 10^x$.

B. Natural Logarithms

Base e logarithms are called **natural logarithms**. They are denoted manually as ln & as LN on a calculator.

The inverse of the natural log, $y = \ln x$, is the exponential function $y = e^x$.

Note: When you see ln x or LN x think log $_{\text{base } e}$ of x ($\log_e x$).

III. Converting Between Logarithmic Form and Exponential Form of an Equation

We can use the definition of a logarithm to rewrite a logarithmic equation such as $y = \log_2 x$ into an equivalent exponential equation $x = 2^y$ and vice versa.

A. Given **y** = log **$_a$ x**

Base **a** raised to the **y** power equals the argument **x**. → $a^y = x$

Example 1

Convert log .01 = –2 into exponential form. (#47 p. 387)

Base **10** raised to the **–2** power equals **.01** → $10^{-2} = .01$

B. Given $a^y = x$

The exponent **y** equals the log base **a** of **x**. → $y = \log_a x$

Example 2

Convert $p^k = 3$ into logarithmic form. (#43 p. 387)

The exponent **k** equals the log base **p** of **3**. → $k = \log_p 3$

IV. Evaluating Logarithms

A. The Domain of a Logarithm

The domain of a logarithm which has not been shifted horizontally is $(0, \infty)$. Thus you cannot take a log of 0 or of a negative number.

B. Some Properties of Logarithms

1. $\log_a 1 = 0$ because $a^0 = 1$
2. $\log_a a = 1$ because $a^1 = a$
3. $\log_a a^x = x$ because $a^{\log_a x} = x$
4. If $\log_a x = \log_a y$, then $x = y$.

Example 3

Find $\ln\sqrt{e}$. Do not use a calculator. (#33 p. 387)

Since $\sqrt{e} = e^{\frac{1}{2}}$ and ln x and e^x are inverse functions, $\ln\sqrt{e} = \ln e^{\frac{1}{2}} = \frac{1}{2}$

C. Evaluating a Logarithm without a Calculator

A logarithm is an exponent. It is the exponent (y) you must raise the base (a) to in order to get the argument (x).

If you are asked to evaluate $\log_a x$ ask yourself the question:

Base **a** raised to what power ☐ equals **x**? → $a^{\square} = x$

Example 4

Find the following. Do not use a calculator. (# 10 and #16 p. 387)

a. $\log_3 9 \rightarrow 3^{\square} = 9 \rightarrow 3^2 = 9 \rightarrow \log_3 9 = 2$

b. $\log_8 2 \rightarrow 8^{\square} = 2 \rightarrow \sqrt[3]{8} = 2 \rightarrow 8^{\frac{1}{3}} = 2 \rightarrow \log_8 2 = \frac{1}{3}$

D. Evaluating a Logarithm Using a Calculator

1. ***Directly***

Our calculator has two built in logarithms. The **common logarithm** (base 10) LOG and the **natural logarithm** (base e) LN. Thus you can evaluate expressions like log(3) or ln(50), directly by using these keys.

Example 5

Find each of the following using a calculator. Round to four decimal places.

a. log(3) = .4771 (#55 p. 387) b. ln(50) = 3.9120 (# 62 p. 387)

2. ***Using the Change of Base Theorem***

To evaluate a logarithm with a base other than 10 or e on the calculator, we must use the Change of Base Theorem. This theorem allows us to rewrite a log base b as a quotient of two logs with base a.

$$\log_b m = \frac{\log_a m}{\log_a b}$$

Although we can change b to any base, it is most practical to change it to either base 10 or base e. Consequently, the most frequently used forms of the Change of Base Theorem are:

$$\log_b m = \frac{\log m}{\log b} \quad \text{and} \quad \log_b m = \frac{\ln m}{\ln b}$$

Example 6

Find $\log_3 7$ using a calculator. Round to four decimal places. (like #70 & 75 on p. 387)

We can say $\log_3 7 = \frac{\ln 7}{\ln 3}$ or $\frac{\log 7}{\log 3} = 1.771$.

V. Transformations of Logarithmic Functions

$y = \log_a(x-h)$ horizontal shift right h **Note**: A horizontal shift changes the VA.

$y = \log_a(x+h)$ horizontal shift left h

$y = -\log_a x$ x-axis reflection

$y = \log_a(-x)$ y-axis reflection

$y = c \cdot \log_a x,\ |c| > 0$ vertical stretch by a factor of c

$y = c \cdot \log_a x,\ 0 < |c| < 1$ vertical shrink by a factor of c

$y = \log_a x + k$ vertical shift up k

$y = \log_a x - k$ vertical shift down k

VI. Graphing Logarithmic Functions

A. <u>Graphing Log Functions without a Calculator</u>

1. Describe the transformations which have occurred to the "plain vanilla" function.
2. Find and graph the vertical asymptote. [Set the argument equal to 0 and solve for x.]
3. Isolate the log expression and then use the definition of a log to convert to exponential form.
4. Find the "elbow" of the graph by setting the exponent = 0 and solving for **y**.
5. Make that **y value** (the y of the "elbow") the center of your t-chart. Plug in two y values below and two y values above.

   Note: It is also often helpful to include the x-intercept in your t-chart.
6. Plot your 5 points and connect them with a smooth curve using the VA as a guide.

Example 7

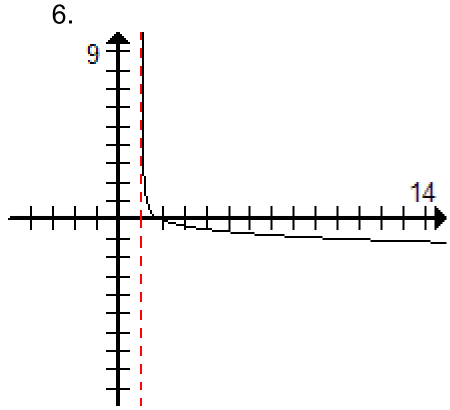
Graph $f(x) = -\frac{1}{2}\log_3(x-1)$ by doing the following:

1) List the base function & transformations. 2) Find the vertical asymptote. 3) Rewrite the function in exponential form. 4) Make a five point t-chart with the "elbow" in the middle.
5) Graph the function.

1. Base function: $y = \log_3 x$
   Transformations: x-axis reflection, vertical shrink by a factor of $\frac{1}{2}$, horizontal shift right 1
2. VA $x - 1 = 0 \rightarrow x = 1$
3. $y = -\frac{1}{2}\log_3(x-1) \rightarrow -2y = \log_3(x-1) \rightarrow 3^{-2y} = x - 1 \rightarrow x = 3^{-2y} + 1$
4. $-2y = 0 \rightarrow y = 0$ (y of the "elbow")
5.

x	y
82	–2
10	–1
2	0
$\frac{10}{9}$	1
$\frac{82}{81}$	2

6.



B. Graphing Log Functions with a Calculator

If the log is base 10 or base e, graph directly using the appropriate key (LOG or LN). If the base is anything other than 10 or e, use the change of base formula to convert to base 10 or e, then graph directly.

VII. *Applications of Logarithms*

A. Earthquake Magnitude $R = \log \frac{I}{I_0}$

R is the magnitude, measured on the Richter scale, of an earthquake of intensity I.
I_0 is the threshold intensity, the slightest tremor that the machine can measure.
If one earthquake is 10 times as intense as another, its magnitude on the Richter scale is 1 greater than that of the other. If one earthquake is 100 times as intense as another, its magnitude on the Richter scale will be 2 higher, and so on.

Example 8

The 1906 earthquake in San Francisco had a magnitude of $10^{8.25}\, I_0$. What was its magnitude on the Richter scale? (#93b p. 388)

$R = \log \frac{10^{8.25} I_0}{I_0} = \log 10^{8.25} = \mathbf{8.25}$

B. ***Acidity or Basicity of a Substance*** $pH = -\log [H^+]$

pH is a measure of the acidity or basicity of a substance. $[H^+]$ is the hydrogen ion concentration of the substance, in moles per liter. Pure water has a pH of 7. Substances which have a pH below 7 are said to be acidic and substances which have a pH above 7 are said to be basic or alkaline.

Example 9

A tomato has a hydrogen ion concentration of 6.3×10^{-5} moles per liter. What is its pH?

$pH = -\log (6.3 \times 10^{-5}) = \mathbf{4.2}$ (#94e p. 388)

Example 10

The pH of a cup of rainwater is 5.4. What is its hydrogen ion concentration? (#95b p. 388)

$5.4 = -\log [H^+] \rightarrow -5.4 = \log [H^+] \rightarrow [H^+] = 10^{-5.4} =$ **4.0×10^{-6} moles per liter**

C. ***Loudness of Sound*** $L = 10 \log \frac{I}{I_0}$

L is the loudness of a sound measured in decibels, I is the intensity of the sound, and I_0 is the threshold intensity – the quietest sound the human ear can detect. If one sound is ten times as intense as another, its loudness is 1 decibel higher than that of the other. If a sound is 100 times as intense as another, its loudness is 2 decibels higher, and so on.

Example 11

A machine gives off a sound with an intensity of $2{,}500{,}000 I_0$. What is the decibel rating of the sound? (#97b p. 389)

$L = 10 \log \frac{2{,}500{,}000\, I_0}{I_0} = 10 \log 2{,}500{,}000 =$ **64 decibel**

D. ***Walking Speed*** $w(P) = .37 \ln P + .05$

w is the average walking speed, in feet per second, of a person living in a city of population *P*, in thousands.

Example 12

According to a 2007 Census, the population of Rogers, Arkansas is 54,959. Find the average walking speed of people living in Rogers. (like #91 p. 388)

Since P is in thousands we substitute 54.959 in for P.

$w(54.959) = .37 \ln 54.959 + .05$ $\approx$ **1.53 feet per second**