4.5 Solving Exponential and Logarithmic Equations

I. Review of Some Key Properties of Exponential and Logarithmic Functions

A. Base-Exponent Property

$a^x = a^y$ if and only if x = y

B. Inverse Properties

Exponential functions and logarithmic functions are inverses which undo one another.

1. $\log_a a^x = x$

2. $a^{\log_a x} = x$

C. Change of Base Formula

$\log_b m = \dfrac{\log m}{\log b}$ or $\log_b m = \dfrac{\ln m}{\ln b}$

D. Property of Logarithmic Equality

$\log_a x = \log_a y$ if and only if x = y.

E. Product Rule of Logarithms

$\log_a (m \cdot n) = \log_a m + \log_a n$

F. Quotient Rule of Logarithms

$\log_a \left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

G. Power Rule of Logarithms

$\log_a (m^p) = p \cdot \log_a m$

II. Exponential Equations

A. Definition

Equations with variables in the exponents are called **exponential equations**.

Examples: $3^{5x} = 27$ $e^{x+1} = 7$ $600 = 300e^{.04t}$

B. Strategies for Solving Exponential Equations

Method 1: Get the Same Base

Rewrite each side of the equation as a power of the same base and then apply the Base-Exponent Property to solve for the variable. To check, plug your un-rounded solution into the original equation and see if it works.

Example 1 $3^{7x} = 81$

$3^{7x} = 81 \rightarrow 3^{7x} = 3^4 \rightarrow 7x = 4 \rightarrow x = \dfrac{4}{7}$

Check: $3^{\left(7 \cdot \frac{4}{7}\right)} = 3^4 = 81$

Method 2: Using $\log_a$ to undo a^x

Isolate the exponential expression. Take the log base a of both sides. Use the inverse property to simplify one side and, if necessary, the change of base formula to simplify the other. Solve for the variable. To check, plug your un-rounded solution into the original equation and see if it works.

Example 2 $2^{x-1}+3=8$

$2^{x-1}+3=8 \rightarrow 2^{x-1}=5 \rightarrow \log_2 2^{x-1} = \log_2 5 \rightarrow x-1=\frac{\ln 5}{\ln 2} \rightarrow$

$x=\frac{\ln 5}{\ln 2}+1 \approx 3.3219$

Check: $2^{(3.321928095-1)}+3=8$

Method 3: Using ln and the Power Rule to undo a^x

Isolate the exponential expression. Take the natural log of both sides. Use the power rule of logarithms to make the exponent a coefficient. Solve for the variable. To check, plug your un-rounded solution into the original equation and see if it works.

Example 3 $3\left(4^{x+2}\right)-5=16$

$3\left(4^{x+2}\right)-5=16 \rightarrow 3\left(4^{x+2}\right)=21 \rightarrow 4^{x+2}=7 \rightarrow \ln 4^{x+2} = \ln 7 \rightarrow$

$(x+2)\ln 4 = \ln 7 \rightarrow x+2=\frac{\ln 7}{\ln 4} \rightarrow x=\frac{\ln 7}{\ln 4}-2 \approx -.5963$

Check: $3\left(4^{(-.596322539+2)}\right)-5=16$

Method 4: Graphical Solution

Put the equation in standard form (everything on the left side of the equal sign and zero on the right). Let the left side equal y_1. Adjust the window. Use the Zero / Root feature to find the zeros.

Example 4 $.082e^{.05x}=.034$

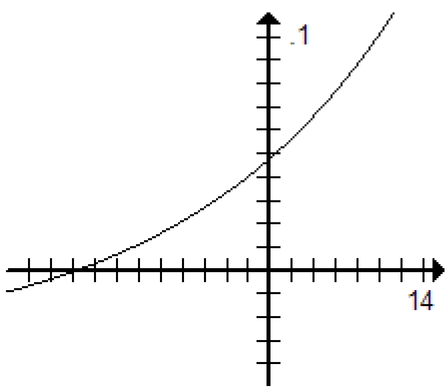
$y_1 = .082e^{.05x} - .034$

Adjust window

2nd Trace 2 (Zero / Root)

$x = -17.607$

Check: $.082e^{(.05\times -17.60717445)}=.034$



III. Logarithmic Equations

A. Definition

Equations containing variables in logarithmic expressions are called **logarithmic equations**.

Examples: $\log_5 x = 9$ $\ln(x+1)+\ln(x-1)=3$ $\log_3(2x-4)=\log_3(x+7)$

B. Strategies for Solving Logarithmic Equations

Method 1: Using the Definition of a Logarithm $y=\log_a x \leftrightarrow x=a^y$

Use the properties of logarithms to write the equation so there is no more than one log on each side. Use the definition of a log to rewrite the equation in exponential form. Solve for the variable. To check, plug your un-rounded solution into the original equation and see if it works. You may have to use the change of base formula.

Example 5 $\log_5(8-7x)=3$

$$\log_5(8-7x)=3 \rightarrow 5^3 = 8-7x \rightarrow 125=8-7x \rightarrow x=-\frac{117}{7}$$

Check: $\log_5\left(8-7\left(-\frac{117}{7}\right)\right) \rightarrow \log_5 125 \rightarrow \log_5 5^3 \rightarrow 3$

or $\log_5 125 \rightarrow \frac{\ln 125}{\ln 5} \rightarrow 3$

Method 2: Exponentiating Both Sides

Use the properties of logarithms to write the equation so there is no more than one log on each side. Exponentiate both sides using base a. Simplify one side using the inverse property to solve for the variable. To check, plug your un-rounded solution into the original equation and see if it works. You may have to use the change of base formula.

NOTE: It is especially important to check your solutions if you end up with a quadratic.

Example 6 $\log_5(x+4)+\log_5 x=2$

$$\log_5(x+4)+\log_5 x = 2 \rightarrow \log_5[(x+4)\cdot x]=2 \rightarrow \log_5\left[x^2+4x\right]=2 \rightarrow$$

$$5^{\log_5\left(x^2+4x\right)} = 5^2 \rightarrow x^2+4x=25 \rightarrow x^2+4x-25=0 \rightarrow$$

$x \approx -7.3852$ or 3.3852

Check: $\log_5(-7.385164807+4)+\log_5(-7.385164807) \rightarrow \textit{doesn't work}$

$\log_5(3.385164807+4)+\log_5(3.385164807) \rightarrow 2$

Method 3: Graphical Solution

Put the equation in standard form (everything on the left side of the equal sign and zero on the right). Let the left side equal y_1. If the logs are not base 10 or base e, use the change of base formula to rewrite the logs. Adjust the window. Use the Zero / Root feature to find the zeros.

Example 7 $2 \ln (x - 5.1) = 8.9$

$y_1 = 2 \ln (x - 5.1) - 8.9$

Adjust window

2nd Trace 2 (Zero / Root)

$x = 90.726$

Check: $2 \ln (90.726944 - 5.1) = 8.9$