7.2 Arithmetic Sequences and Series

I. Arithmetic Sequences

In an **arithmetic sequence** each term after the first is found by adding the same number to the preceding term.

The number which we add to a term to get the next term is called the **common difference**, d.

In an arithmetic sequence, $a_{n+1} = a_n + d$ for any integer $n \geq 1$.

To find the common difference, d, we choose any term beyond the first and subtract the preceding term from it.

Example 1

The sequence 3, 8, 13, 18, 23, 28, … is an arithmetic sequence because we can add 5 to any term to produce the next term.

The sequence 1, 1, 2, 3, 5, 8, 13, … is not an arithmetic sequence because there is no common difference between the terms.

Example 2

Is the following sequence arithmetic? If it is, identify the first term and the common difference.
$\frac{2}{3}, \frac{17}{12}, \frac{13}{6}, \frac{35}{12}, \ldots$

workspace

With fractions it might be easier to see a pattern if we write the terms with a common denominator.
$\frac{8}{12}, \frac{17}{12}, \frac{26}{12}, \frac{35}{12}, \ldots$ Now we can see that consecutive terms differ by $\frac{9}{12}$.

conclusion

Yes, this sequence is arithmetic. The first term is $\frac{2}{3}$ and the common difference is $d = \frac{3}{4}$.

II. n^{th} Term of an Arithmetic Sequence

The **n^{th} term** of an arithmetic sequence is given by: $a_n = a_1 + (n - 1) \cdot d$ for any integer $n \geq 1$.

This formula has four variables, a_n, a_1, n, and d. If we know any three of the variables, we can find the fourth.

Example 3

Find the fifteenth term of the arithmetic sequence 7, 3, –1, –5, –9, …

We want to find a_{15}. We know $a_1 = 7$, $n = 15$, and $d = 3 - 7 = -4$. Therefore,

$a_{15} = 7 + (15 - 1) \cdot (-4) = 7 + 14(-4) = 7 - 56 = \mathbf{-49}$

Example 4

In the sequence 3, $\frac{7}{3}$, $\frac{5}{3}$, … what term is $-\frac{35}{3}$?

We want to find n. We know $a_n = -\frac{35}{3}$, $a_1 = 3$, and $d = \frac{5}{3}-\frac{7}{3}=-\frac{2}{3}$. Therefore,

$-\frac{35}{3} = 3 + (n-1)\cdot\left(-\frac{2}{3}\right) \rightarrow -\frac{35}{3} = 3 - \frac{2}{3}n+\frac{2}{3} \rightarrow -\frac{35}{3} = \frac{11}{3}-\frac{2}{3}n \rightarrow$

$-\frac{35}{3}-\frac{11}{3} = -\frac{2}{3}n \rightarrow -\frac{46}{3}=-\frac{2}{3}n \rightarrow n = \left(-\frac{46}{3}\right)\left(-\frac{3}{2}\right) = 23$

conclusion: $-\frac{35}{3}$ is the 23rd term in the sequence.

Example 5 (#23 p. 592)

In an arithmetic sequence, $a_{17} = \frac{25}{3}$ and $a_{32} = \frac{95}{6}$. Find a_1 and d. Then write the first 5 terms of the sequence.

Let's plug in what we know to our formula $a_n = a_1 + (n-1)\cdot d$.

$\frac{25}{3} = a_1 + (17-1)\cdot d \rightarrow \frac{25}{3} = a_1 + 16d$ AND $\frac{95}{6} = a_1 + (32-1)\cdot d \rightarrow \frac{95}{6} = a_1 + 31d$

Now we can solve the system of equations $a_1 + 16d = \frac{25}{3}$

$a_1 + 31d = \frac{95}{6}$ to find a_1 and d.

Solving for a_1 in the first equation and subbing into the second, we get

$a_1 = -16d + \frac{25}{3} \rightarrow -16d + \frac{25}{3} + 31d = \frac{95}{6} \rightarrow 15d = \frac{95}{6} - \frac{25}{3} = \frac{95}{6} - \frac{50}{6} = \frac{45}{6} \rightarrow$

$15d = \frac{15}{2} \rightarrow d = \frac{15}{2}\cdot\frac{1}{15}=\frac{1}{2}$ $a_1 = -16(\frac{1}{2}) + \frac{25}{3} = -8 + \frac{25}{3} = -\frac{24}{3}+\frac{25}{3} = \frac{1}{3}$

Since $a_1 = \frac{1}{3}=\frac{2}{6}$ and $d = \frac{1}{2}=\frac{3}{6}$, the first five terms of the sequence are $\frac{2}{6}$, $\frac{5}{6}$, $\frac{8}{6}$, $\frac{11}{6}$, $\frac{14}{6}$

or, in reduced form, $\frac{1}{3}$, $\frac{5}{6}$, $\frac{4}{3}$, $\frac{11}{6}$, $\frac{7}{3}$.

III. Sum of the First n Terms of an Arithmetic Sequence

The sum of the first n terms of an arithmetic sequence is given by $S_n = \frac{n}{2}(a_1 + a_n)$.

Example 6

Find the sum of the first 20 terms of the sequence –3, 2, 7, 12, ….

Before we can find S_{20}, we must first find d and then a_{20}.

$d = 12 - 7 = 5$ $a_{20} = -3 + (20-1)\cdot 5 = -3 + 19(5) = -3 + 95 = 92$

Therefore, $S_{20} = \frac{20}{2}(-3 + 92) = 10(89) = 890$

Example 7

Find the sum $\sum_{n=7}^{12} \frac{5-3n}{4}$.

<u>Method 1</u>

Plug 7 through 12 into the defining expression and then add the resulting terms.

$$\sum_{n=7}^{12} \frac{5-3n}{4} = \frac{5-3(7)}{4} + \frac{5-3(8)}{4} + \frac{5-3(9)}{4} + \frac{5-3(10)}{4} + \frac{5-3(11)}{4} + \frac{5-3(12)}{4} =$$

$$-\frac{16}{4} - \frac{19}{4} - \frac{22}{4} - \frac{25}{3} - \frac{28}{4} - \frac{31}{4} = -\frac{141}{4}$$

<u>Method 2</u>

Use the summation formula for arithmetic sequences $S_n = \frac{n}{2}(a_1 + a_n)$.

$n = 6$ $\quad a_1 = a_7 = \frac{5-3(7)}{4} = \frac{5-21}{4} = \frac{-16}{4} = -4$ $\quad a_6 = a_{12} = \frac{5-3(12)}{4} = \frac{5-36}{4} = \frac{-31}{4}$

$$S = \frac{6}{2}\left(-4 - \frac{31}{4}\right) = 3\left(-\frac{16}{4} - \frac{31}{4}\right) = 3\left(-\frac{47}{4}\right) = -\frac{141}{4}$$

IV. Applications

The translation of some applications or problem-solving situations may involve arithmetic sequences or series.

Example 8

Theaters are often built with a decreasing number of seats per row as you move from the back of the theater towards the front. Since there are fewer front row seats, they are sold at a premium.

If a theater has 28 seats in the first row, 32 in the second, 36 in the third, and so on for 20 rows, how many seats are in the theater altogether?

$a_1 = 28$ $\quad d = 36 – 32 = 4$ $\quad n = 20$ $\quad a_{20} = 28 + (20 – 1)\cdot 4 = 28 + 19(4) = 28 + 76 = 104$

$S_{20} = \frac{20}{2}(28 + 104) = 10(132) =$ **1320 seats**