4.1 Slack Variables and the Pivot

I. Introduction

In Chapter 4 we will learn how to solve linear programming problems algebraically using the **simplex method**. This will allow us to solve optimization problems involving more than two variables. To facilitate the solution of problems with a large number of variables, we will label our variables with subscripts \((x_1, x_2, x_3, \text{ etc.})\) rather than using \(x, y, z, \text{ etc.}\).

In section 4.1, we will learn some of the terminology and notation associated with the simplex method, as well as how to set the problems up and read the solution from the final **tableau** (matrix).

II. Standard Maximization Problems

The first type of problem we will look at are **standard maximization problems**. To be a standard maximization problem, a linear programming problem must satisfy the following conditions.

1. The objective function is to be maximized.
2. All variables are non-negative. This condition is met by the inclusion of the **non-negativity constraints**, \(x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \text{ etc.}\).
3. All **structural constraints** must be stated in the form \(a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq b\) with \(b \geq 0\).

III. Slack Variables

The first step in the simplex method is to convert the structural constraints (all constraints other than the non-negativity constraints) from linear inequalities to linear equations. We do this by adding a **slack variable** \((s_1, s_2, s_3, \text{ etc.})\) and replacing the inequality with an equal sign. The slack variable "takes up the slack" of the inequality and represents the amount by which the inequality fails to equal the constant. By their very nature, slack variables must be non-negative, i.e. \(s_n \geq 0\). A different slack variable must be used for each structural constraint.

**Example 1**

Convert the inequality \(2.3x_1 + 5.7x_2 + 1.8x_3 \leq 17\) into an equation by adding a slack variable.

\[2.3x_1 + 5.7x_2 + 1.8x_3 + s_1 = 17\]

**Example 2**

Given the following standard maximization problem, a) determine the number of slack variables needed, b) name them, and c) use slack variables to convert each constraint into a linear equation.

Maximize \(z = 8x_1 + 3x_2 + x_3\)

subject to: \[7x_1 + 6x_2 + 8x_3 \leq 118\]
\[4x_1 + 5x_2 + 10x_3 \leq 220\]
with \(x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\).

a. 2 slack variables are needed since there are two structural constraints
b. The names of the slack variables will be \(s_1\) and \(s_2\).
c. \[7x_1 + 6x_2 + 8x_3 + s_1 = 118\]
\[4x_1 + 5x_2 + 10x_3 + s_2 = 220\]

IV. Rewriting the Objective Function

The second step in the simplex method is to rewrite the objective function so that all the variables are on the left side of the equal sign and the constant is on the right. When we do this, the coefficient of the variable we are optimizing must be positive.

**Example 3**

Rewrite the objective function from Example 2 so that all variables are on the left and the constant is on the right.

\[-8x_1 - 3x_2 - x_3 + z = 0\]
V. The Initial Simplex Tableau

The augmented matrix formed from the slack equations and the rewritten objective function is called the initial simplex tableau. Tableau is French for matrix. In the initial tableau, the rewritten objective function is written on the bottom and the variables are lined up vertically. Zeros are inserted as placeholders for missing variables. Each column is labeled with the appropriate variable. The coefficients of the rewritten objective function (excluding the 1 and zero on the far right) are called indicators.

Example 4

Introduce slack variables as necessary, then write the initial simplex tableau for the following linear programming problem.

Find \( x_1 \geq 0 \) and \( x_2 \geq 0 \) such that

\[
\begin{align*}
x_1 + x_2 &\leq 10 \\
5x_1 + 2x_2 &\leq 20 \\
x_1 + 2x_2 &\leq 36
\end{align*}
\]

and \( z = x_1 + 3x_2 \) is maximized.

\[
\begin{align*}
x_1 + x_2 &\leq 10 & \rightarrow & x_1 + x_2 + s_1 & = 10 \\
5x_1 + 2x_2 &\leq 20 & \rightarrow & 5x_1 + 2x_2 + s_2 & = 20 \\
x_1 + 2x_2 &\leq 36 & \rightarrow & x_1 + 2x_2 + s_3 & = 36 \\
z = x_1 + 3x_2 & \rightarrow & -x_1 - 3x_2 + z & = 0
\end{align*}
\]

VII. Basic and Non-basic Variables

In any simplex tableau, the variables can be broken into two groups – basic variables and non-basic variables. Variables in columns containing only one nonzero entry above the indicator row are called basic variables. The value of a basic variable is found by dividing its nonzero entry into the constant at the end of its row. Variables in columns containing more than one nonzero entry above the indicator row are called non-basic variables. The value of any non-basic variable is always zero. The variable being optimized is not considered basic or non-basic. Its value is found by dividing its nonzero entry into the constant at the end of its row.

Example 5

In the matrix from Example 4, the basic variables are \( s_1, s_2, \) and \( s_3 \). The non-basic variables are \( x_1 \) and \( x_2 \).

VII. Basic Solutions, Basic Feasible Solutions, and Optimal Solutions

A solution found by setting the non-basic variables equal to zero and solving for the basic variables is called a basic solution. Basic solutions are related to the intersection points of the extended boundary lines of the feasible region. If a basic solution has no negative values, it is called a basic feasible solution. A basic feasible solution corresponds to a corner point of the feasible solution. The optimal solution is obtained when all variables are non-negative and there are no negatives in the indicator row.

Example 6

Write the solution that can be read from the following simplex tableau.

Non-basic variables: \( x_1, x_2, s_1 \)

Basic variables: \( x_3, s_2, s_3 \)

\[
\begin{align*}
2x_3 & = 16 \quad \rightarrow \quad x_3 = 8; \quad s_2 = 6 \\
5s_3 & = 35 \quad \rightarrow \quad s_3 = 7; \quad 3z = 36 \quad \rightarrow \quad z = 12
\end{align*}
\]

Basic feasible solution: \( z = 12 \) when \( x_1 = 0, x_2 = 0, x_3 = 8, s_1 = 0, s_2 = 6, s_3 = 7 \)
VIII. Pivoting

To find the optimal solution of a linear programming, we use two row operations to change the initial simplex tableau into the final tableau. These two row operations are Multiply and Pivot.

To multiply involves multiplying the elements of a row by a nonzero number. We multiply by the reciprocal of the number occupying the pivot position to get a one in that position.

To pivot involves adding a multiple of one row to another row. We do this to get a zero above or below a pivot position. The row we take a multiple of is called the pivot row. The pivot or pivot position is the key entry in the pivot row, the entry we use to get the zeros. We will talk about how to select a pivot in section 4.2. In this section we just want to practice the pivoting operation.

Pivoting clears a column of all nonzero non-pivot entries. If you have a one in the pivot position and an A in the position you want to clear, the pivoting operation is: \(\text{cp } R_p + A \cdot R_c\) where \(R_p\) is the pivot row and \(R_c\) is the row you are getting a zero in. If you have a B in the pivot position and an A in the position you want to clear, the pivoting operation is: \(-A \cdot R_p + B \cdot R_c\).

Note: Since we will use the ROWOPS program to solve simplex problems, we will always get a one in the pivot position before we pivot. In doing this, our intermediate matrices may differ from those of someone who pivots first and gets ones later, but our final solutions will be the same.

Example 7

Pivot once as indicated in the simplex tableau. Read the solution from the result.

\[
\begin{bmatrix}
  x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \\
  2 & 2 & 3 & 1 & 0 & 0 & 0 & 500 \\
  4 & 1 & 1 & 0 & 1 & 0 & 0 & 300 \\
  7 & 2 & 4 & 0 & 0 & 1 & 0 & 700 \\
\end{bmatrix}
\]

The grey box indicates that the pivot is in row 1 column 2. Since there is a 2 occupying this position, we will first multiply row 1 by \(\frac{1}{2}\) (the reciprocal of 2) to get a one in the pivot position. On paper we write this row operation as \(\frac{1}{2}R_1\).

Manual Workspace:

\[
\frac{1}{2}R_1 = \frac{1}{2}(2 \ 2 \ 3 \ 1 \ 0 \ 0 \ 0 \ 500) = 1 \ 1 \ \frac{3}{2} \ \frac{1}{2} \ 0 \ 0 \ 0 \ 250 \ \text{new } R_1
\]

Note: You may use the Rowops program to perform all row operations on matrices. You do not have to show your manual arithmetic. I have shown my manual work here as a reference for students who may not have a graphing calculator with the Rowops program or who may choose to work manually.

\[
\begin{bmatrix}
  x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \\
  2 & 2 & 3 & 1 & 0 & 0 & 0 & 500 \\
  4 & 1 & 1 & 0 & 1 & 0 & 0 & 300 \\
  7 & 2 & 4 & 0 & 0 & 1 & 0 & 700 \\
\end{bmatrix}
\]

Since there is a 1 in \(R_2C_2\), to turn this into a zero, we will multiply the pivot row \((R_1)\) by the opposite number \((-1)\) and add that result to the row we getting a zero in \((R_2)\). On paper we write this row operation as \(-1R_1 + R_2\).

Since there is a 2 in \(R_3C_2\), to turn this into a zero, we will multiply the pivot row \((R_1)\) by the opposite number \((-2)\) and add that result to the row we getting a zero in \((R_3)\). On paper we write this row operation as \(-2R_1 + R_3\).
Since there is a –4 in \( R_4C_2 \), to turn this into a zero, we will multiply the pivot row \( (R_1) \) by the opposite number \( (4) \) and add that result to the row we getting a zero in \( (R_4) \). On paper we write this row operation as \( 4R_1 + R_4 \).

**Manual Workspace:**

\[
\begin{array}{cccccccc}
-1 & -1 & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 & -250 \\
+ R_2 & 4 & 1 & 1 & 0 & 1 & 0 & 300 \\
= & 3 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 & 50 \\
\text{new } R_2 & & & & & & & \\
-2R_1 & -2 & -2 & -3 & -1 & 0 & 0 & 0 & -500 \\
+ R_2 & 7 & 2 & 4 & 0 & 0 & 1 & 0 & 700 \\
= & 5 & 0 & 1 & -1 & 0 & 1 & 0 & 200 \\
\text{new } R_3 & & & & & & & \\
4R_1 & 4 & 4 & 6 & 2 & 0 & 0 & 0 & 1000 \\
+ R_4 & -3 & -4 & -2 & 0 & 0 & 0 & 1 & 0 \\
= & 1 & 0 & 4 & 2 & 0 & 0 & 1 & 1000 \\
\text{new } R_4 & & & & & & & \\
\end{array}
\]

Basic variables: \( x_2, s_2, s_3 \)  
Non-basic variables: \( x_1, x_3, s_1 \)

Basic Feasible Solution: Maximum \( z = 1000 \) when  
when \( x_1 = 0, x_2 = 250, x_3 = 0, s_1 = 0, s_2 = 50, s_3 = 200 \)

### Example 8

Set up the following problem for solution by the simplex method. First express the linear constraints and objective function, then add slack variables, and then set up the initial simplex tableau.

A manufacturer of bicycles builds racing, touring, and mountain models. The bicycles are made of both aluminum and steel. The company has available 91,800 units of steel and 42,000 units of aluminum. The racing, touring, and mountain models need 17, 27, and 34 units of steel, and 12, 21, and 15 units of aluminum, respectively. How many of each type of bicycle should be made in order to maximize profit if the company makes \$8 per racing bike, \$12 per touring bike, and \$22 per mountain bike? What is the maximum profit?

\( x_1 \) = number of racing bikes  \( x_2 \) = number of touring bikes  \( x_3 \) = number of mountain bikes

**Profit:** \( P = 8x_1 + 12x_2 + 22x_3 \)

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>17</td>
<td>27</td>
<td>34</td>
</tr>
<tr>
<td>Aluminum</td>
<td>12</td>
<td>21</td>
<td>15</td>
</tr>
</tbody>
</table>

Maximize profit \( P = 8x_1 + 12x_2 + 22x_3 \) subject to  
\( 17x_1 + 27x_2 + 34x_3 \leq 91,800 \) \( 12x_1 + 21x_2 + 15x_3 \leq 42,000 \)  
with \( x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \)

\[
\begin{array}{cccccc}
17 & 27 & 34 & x_1 & s_1 & = 91,800 \\
12 & 21 & 15 & x_1 & s_2 & = 42,000 \\
P = 8x_1 + 12x_2 + 22x_3 & \rightarrow & -8x_1 - 12x_2 - 22x_3 & + P & = 0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
x_1 & x_2 & x_3 & s_1 & s_2 & P \\
17 & 27 & 34 & 1 & 0 & 0 & 91800 \\
12 & 21 & 15 & 0 & 1 & 0 & 42000 \\
-8 & -12 & -22 & 0 & 0 & 1 & 0 \\
\end{array}
\]