5.2 Future Value of an Annuity

I. Ordinary Annuities

A. Terminology

1. An **annuity** is a sequence of equal payments made at equal periods of time.
2. With an **ordinary annuity**, payments are made at the end of the time period and the frequency of payments is the same as the frequency of compounding.
3. The **payment period** is the time between payments.
4. The **term** of the annuity is the time from the beginning of the first payment period to the end of the last period.
5. The **future value of the annuity**, the final sum on deposit, is the sum of all the payments and all the compounded interest.

B. Formula for the Future Value of an Ordinary Annuity

\[
S = R \left[ \frac{\left(1 + \frac{r}{m}\right)^{m \cdot t} - 1}{\frac{r}{m}} \right]
\]

or

\[
S = R \left[ \frac{\left(1 + \frac{i}{m}\right)^{n} - 1}{\frac{i}{m}} \right]
\]

S = the future value
R = the payment
r = the annual interest rate
m = the number of times interest is compounded per year
t = time in years
i = the periodic interest rate; \(i = \frac{r}{m}\)
n = the total number of compoundings during the term of the annuity; \(n = m \cdot t\)

**Note:** Amount of the future value from contributions = \(R \cdot m \cdot t\)

Amount of the future value from interest = \(S - (R \cdot m \cdot t)\)

**Example 1**

Find the future value of the following ordinary annuity, if payments are made and interest is compounded as given. Then determine how much of this value is from contributions and how much is from interest.

\[ R = 4600; \ 8.73\% \text{ interest compounded quarterly for } 9 \text{ years} \]

\[
S = 4600 \left[ \frac{\left(1 + \frac{.0873}{4}\right)^{4 \cdot 9} - 1}{\frac{.0873}{4}} \right] = 247,752.70
\]

Amount from contributions = \(4600 \times 4 \times 9 = 165,600\)

Amount from interest = \(247,752.70 - 165,600 = 82,152.70\)
Note: When entering an annuity set-up into your calculator manually, use parentheses exactly as shown in the formulas in these notes.

II. Sinking Funds
A. Introduction
A sinking fund is a fund set up to receive periodic payments. The periodic payments, together with the interest earned by the payments, are designed to produce a given sum at some time in the future. If the payments are all the same and they are made at the end of a regular time period, they form an ordinary annuity.

B. Formula for a Sinking Fund Payment
\[ R = \frac{(S \cdot \frac{r}{m})}{\left(1 + \frac{r}{m}\right)^{m \cdot t} - 1} \] or \[ R = \frac{(S \cdot i)}{(1 + i)^n - 1} \]

Example 2
Find the amount of each payment to be made into a sinking fund so that enough will be present to accumulate the following amount. Payments are made at the end of each period.
$65,000; money earns 7.5% compounded quarterly for \( 2 \frac{1}{2} \) years

\[ R = \frac{(65,000 \cdot .075/4)}{(1 + .075/4)^{(4 \cdot 2.5)} - 1} = \$5,970.23 \]

III. Annuities Due
A. Introduction
An annuity in which payments are made at the beginning of each time period is called an annuity due. To find the future value of an annuity due, we treat each payment as if it were made at the end of the preceding period. We calculate the future value of an ordinary annuity for the number of payment periods plus one extra, and then we subtract the amount of one payment.

B. Formula for the Future Value of an Annuity Due
\[ S = R \left[ \left(1 + \frac{r}{m}\right)^{m \cdot t + 1} - 1 \right] \] or \[ S = R \left( \frac{1 + (1 + i)^{n + 1}}{i} - 1 \right) - R \]

Example 3
Find the future value of the following annuity due. Assume that interest is compounded annually.
R = $1700; \ i = .04; \ n = 15
\[ S = 1700 \left(1 + .04\right)^{15+1} \frac{1}{.04} - 1700 = \$35,401.70 \]
On all financial problems you must show your setup (the appropriate formula with the values plugged in for the variables) and your final answer. You may use the TVM Solver to do the calculations, or you may do the arithmetic manually on your calculator. Round all monetary values to two decimal places.

IV. Solving Finance Problems Using the TVM Solver on a TI-83/84

1. On a TI-83, hit 2nd x–1 and select 1: TVM Solver.
On a TI-83/84 Plus, hit APPS, select 1: Finance, and then select 1: TVM Solver.

2. Enter the appropriate information. Initially, enter 0 for the variable you are solving for.
N = total number of payments; N = m · t
I% = the annual interest rate in percentage form; **Note:** Enter 8.25% as 8.25 not as .0825.
PV = present value
PMT = payment; **Note:** Money coming out of your pocket is entered as a negative.
FV = future value
P/Y = payments per year
C/Y = compoundings per year; **Note:** P/Y and C/Y will always be the same
PMT: END BEGIN For an ordinary annuity, set PMT to END. For an annuity due, set PMT to BEGIN

3. Once all the data is entered, place the cursor at the variable you are solving for and hit **ALPHA** ENTER.

Example 4

A 45-year old man puts $2,500 in a retirement account at the end of each quarter until he reaches the age of 60, then makes no further deposits. If the account pays 6% interest compounded quarterly, how much will be in the account when the man retires at age 65? How much of this is interest? [53]

We must do this problem in two parts. First we must calculate the future value of the ordinary annuity after 15 years, when he reaches age 60. Then we will calculate the future value of this amount after 5 years of compounding.

Part 1:

$$ S = 2500 \left[ \frac{\left(1 + \frac{.06}{4}\right)^{4 \cdot 15} - 1}{\frac{.06}{4}} \right] = $240,536.63 $$

Using the TVM Solver:
N = 4 × 15 = 60; I% = 6; PV = 0; PMT = –2500; FV = what we are solving for; P/Y = 4; C/Y = 4; PMT: END
Part 2:

\[ A = 240536.63 \left(1 + \frac{0.06}{4}\right)^{(4 \times 5)} = 323,967.96 \]

Using the TVM Solver:

\[ N = 4 \times 5 = 20; \ I\% = 6; \ PV = -240,536.63; \ PMT = 0; \ FV = \text{what we are solving for}; \]
\[ P/Y = 4; \ C/Y = 4; \ PMT: \text{END} \]

Amount from contributions = 2500 \times 4 \times 15 = 150,000

Amount from interest = 323,967.96 – 150,000 = 173,967.96

Example 5

Harv, the owner of Harv’s Meats, knows that he must buy a new deboner machine in 4 years. The machine costs $12,000. In order to accumulate enough money to pay for the machine, Harv decides to deposit a sum of money at the end of each 6 months in an account paying 6% compounded semiannually. How much should each payment be?

\[ R = \frac{\left( \frac{12000 \cdot 0.06}{2} \right)}{\left(1 + \frac{0.06}{2} \cdot 4\right) - 1} = 1349.48 \]

Using the TVM Solver:

\[ N = 2 \times 4 = 8; \ I\% = 6; \ PV = 0; \ PMT = \text{what we are solving for}; \ FV = -12,000; \]
\[ P/Y = 2; \ C/Y = 2; \ PMT: \text{END} \]