5.3 Present Value of an Annuity; Amortization

I. Present Value of an Ordinary Annuity

A. Introduction

The present value of an annuity is the amount that would have to be deposited in one lump sum today, at the same compound interest rate, in order to produce exactly the same balance at the end of \( t \) years.

B. Formula for the Present Value of an Ordinary Annuity

\[
P = R \left[ \frac{1 - \left(1 + \frac{r}{m}\right)^{-mt}}{\frac{r}{m}} \right] \quad \text{or} \quad P = R \left[ \frac{1 - (1 + i)^{-n}}{i} \right]
\]

Example 1

Find the present value of the following ordinary annuity. [12]

Payments of $50,000 quarterly for 10 years at 4% compounded quarterly

\[
P = 50000 \left[ \frac{1 - \left(1 + \frac{0.04}{4}\right)^{-4 \times 10}}{\frac{0.04}{4}} \right] = \$1,641,734.31
\]

Using the TVM Solver:

\( N = 4 \times 10 = 40; \quad I\% = 4; \quad PV = \text{what we are solving for}; \quad PMT = -50000; \quad FV = 0; \quad P/Y = 4; \quad C/Y = 4; \quad PMT: END \)

Note: When using the TVM Solver, we do not enter \( N \) as a negative.

II. Amortization

A. Introduction

A loan is amortized if both the principal and the interest are paid by a sequence of equal periodic payments.

B. Formula for an Amortization Payment

\[
R = \frac{P}{\left(1 - \left(1 + \frac{r}{m}\right)^{-mt}\right)} \quad \text{or} \quad R = \frac{P \cdot \frac{r}{m}}{\left(1 - \left(1 + \frac{r}{m}\right)^{-mt}\right)}
\]

\[
R = \frac{P}{\left(1 - (1 + i)^{-n}\right)} \quad \text{or} \quad R = \frac{P \cdot i}{\left(1 - (1 + i)^{-n}\right)}
\]

Total of payments = \( R \cdot n \)

Total interest paid = \( (R \cdot n) - P \)
Example 2

Find the payment necessary to amortize the following loan. Then calculate the total of the payments and the total amount of interest paid. \[20\]

$140,000; \ 8\% \text{ compounded quarterly}; \ 15 \text{ quarterly payments} \[54\]

\[i = \frac{.08}{4} = .02\]

\[R = \frac{(140000 \times .02)}{(1 - (1 + .02)^{-15})} = 10,895.57\]

Using the TVM Solver:

\[N = 15; \ i\% = 8\%; \ PV = -140000; \ PMT = \text{what we are trying to find}; \ FV = 0; \]

\[P/Y = 4; \ C/Y = 4; \ PMT:\text{END}\]

Total of payments = $10,895.57 \times 15 = 163,433.55

Total interest paid = 163,433.55 – 140,000 = 23,433.55

C. An Amortization Table

Although equal payments are made to amortize a loan, the loan balance does not decrease in equal steps. An amortization table lists how much of each payment is interest, how much goes to reduce the balance, and how much is owed after each payment.

D. How to Create an Amortization Table by Hand

1. Find the amount financed, \(P = \text{Total principal} - \text{Down payment}\).

2. Find the periodic interest rate, \(i = \frac{r}{m}\), and the total number of payments, \(n = m \times t\).

3. Calculate the payment, \(R = \frac{(P \times i)}{(1 - (1 + i)^{-n})}\).

4. Calculate the interest for the \(n\)th period, \(I_n = i \times P_{n-1}\).

Note: \(P_{n-1}\) is the principal at the end of the previous period

5. Calculate the portion to principal for the \(n\)th payment, \(PP_n = R - I_n\)

6. Calculate the principal at the end of the \(n\)th period, \(P_n = P_{n-1} - PP_n\)

7. Repeat steps 4 – 6 for each payment.

Example 3

Prepare an amortization table for the first four payments of the following loan. \[44\]

One retailer charges $1048 for a laptop computer. A firm of tax accountants buys 8 of these laptops. They make a down payment of $1200 and agree to amortize the balance with monthly payments at 6% compounded monthly for 4 years.

1. \(P = 8(1048) - 1200 = 7184\)

2. \(i = \frac{.06}{12} = .005\) \(n = 12 \times 4 = 48\)

3. \(R = \frac{(7184 \times .005)}{(1 - (1 + .005)^{-48})} = 168.72\)

4. \(I_1 = .005 \times 7184 = 35.92\)
5. \( PP_1 = 168.72 - 35.92 = \$132.80 \)
6. \( P_1 = 7184 - 132.80 = \$7051.20 \)
7a. \( I_2 = .005 \times 7051.20 = \$35.26 \)
b. \( PP_2 = 168.72 - 35.26 = \$133.46 \)
c. \( P_2 = 7051.20 - 133.46 = \$6917.74 \)
8a. \( I_3 = .005 \times 6917.74 = \$34.59 \)
b. \( PP_3 = 168.72 - 34.59 = \$134.13 \)
c. \( P_3 = 6917.74 - 134.13 = \$6783.61 \)
9a. \( I_4 = .005 \times 6783.61 = \$33.92 \)
b. \( PP_4 = 168.72 - 33.92 = \$134.80 \)
c. \( P_4 = 6783.61 - 134.80 = \$6648.81 \)

<table>
<thead>
<tr>
<th>Payment Number</th>
<th>Amount of Payment</th>
<th>Interest for Period</th>
<th>Portion to Principal</th>
<th>Principal at End of Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
<td>$7184.00</td>
</tr>
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<td>$35.92</td>
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<td>4</td>
<td>$168.72</td>
<td>$33.92</td>
<td>$134.80</td>
<td>$6648.81</td>
</tr>
</tbody>
</table>

E. Formula for Finding the Approximate Unpaid Balance After \( x \) Payments

\[
UB = R \left[ \frac{1 - (1 + i)^{- (n-x)}}{i} \right]
\]

Example 4
a. For the loan described in Example 3, find the unpaid balance after 20 payments.

\[
UB = 168.72 \left[ \frac{1 - (1 + .005)^{- (48 - 20)}}{.005} \right] = \$4,398.14
\]

b. How many payments will remain when half of the balance has been paid off?

\[
3592 = 168.72 \left[ \frac{1 - (1 + .005)^{- (48 - x)}}{.005} \right]
\]

We will solve this equation graphically. Set the window to \([0, 48, 3] \times [0, 8000, 400]\)

Enter the left side of the equations as \( y_1 \): \( y_1 = 168.72 \left[ \frac{1 - (1 + .005)^{- (48 - x)}}{.005} \right] \)

Enter the right side of the equation as \( y_2 \): \( y_2 = 3592 \)

Use \( 2^{nd} \) Trace 5:intersect to find the \( x \)-value of the point of intersection.
\( x = 25.433498 \approx 26 \rightarrow 48 - 26 = \text{22 payments remain} \)