8.1 The Multiplication Principle; Permutations

I. Multiplication Principle (Blue box p. 432)
The multiplication principle is a very powerful and general principle that always gives the total number of outcomes that are possible whenever each of these outcomes results from making successive selections.

If an outcome consists of \( n \) successive selections where there are \( m_1 \) choices for the first selection, \( m_2 \) choices for the second selection, \( \ldots \), and \( m_n \) choices for the \( n \)th selection, the total number of outcomes possible is the product \( m_1 \cdot m_2 \cdot m_3 \cdot \ldots \cdot m_n \).

Example: #13 3 selections must be made: plan, style, finish.
For selection 1 (plan) there are 6 choices.
For selection 2 (style) there are 3 choices.
For selection 3 (finish) there are 2 choices.
\[ 6 \cdot 3 \cdot 2 = 36 \] different types of homes are available.

II. Factorial Notation (Blue box p. 434)
For any natural number \( n \), the product of the first \( n \) natural numbers is called \( n \) factorial and is denoted by \( n! \).
\[ n! = n(n-1)(n-2) \ldots (3)(2)(1) \]
Example: \( 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \).
Note: \( n! \) also can be expressed: \( n! = n \cdot (n-1)! \).
Zero factorial, denoted \( 0! \), equals 1. \( 0! = 1 \).
Note: Many calculators have a \( [n!] \) key.
Ex: 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120

Using the fact that \( n! = n(n-1)! \):
\[
\frac{7!}{6!} = \frac{7 \cdot 6!}{6!} = 7
\]

Caution: We cannot cancel unlike factorials: \( \frac{6!}{3!} \neq 2! \)

III. PERMUATIONS
A. Definition
A permutation of a set of distinct objects is an arrangement of the objects in a specific order without repetition. Each rearrangement of the objects is a different permutation.

B. Number of Permutations of \( n \) Objects Taken \( r \) at a Time
The number of permutations of \( n \) distinct objects taken \( r \) at a time without repetition is given by
\[
P(n, r) = \frac{n!}{(n-r)!} \quad \text{or} \quad n(n-1)(n-2)\ldots(n-r+1) \quad \text{for} \quad 0 \leq r \leq n
\]

Note: \( P(n, r) \) may also be written \( P_n^r \) or \( nPr \) or \( P^n_r \).
Most calculators have a key for finding permutations.

Caution: The letter \( P \) here represents permutations, not probability.
(Notice that for permutations there are \( n \) numbers in the \( ( ) \).

C. Distinguishable Permutations
If the \( n \) objects in a permutation are not all distinguishable—that is, if there are \( n_1 \) of type 1, \( n_2 \) of type 2, and so on for \( r \) different types, then the number of distinguishable permutations is:
\[
\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_r!}
\]

Ex: How many distinguishable permutations of the letter in the word intelligent?
11 letters: 2I; 2L; 2n; 2e; 2t; 1l
\[
\frac{11!}{2! \cdot 2! \cdot 2! \cdot 2! \cdot 1!} = 1,247,000
\]
EX: # 5  \( P(13, 2) = \frac{13!}{(13-2)!} = \frac{13!}{11!} = \frac{13 \cdot 12 \cdot 11!}{11!} = 13 \cdot 12 = 156 \)

LIKE 7 or 10 monkeys

EX: # 32  \( P(10, 7) = \frac{10!}{(10-7)!} = \frac{10!}{3!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = \underline{604,800} \)

Using a calculator: \( \begin{array}{c}
10 \ \underline{n!} \ \div \ 3 \ \underline{n!} \ \equiv 3 \ \underline{n!} \ \equiv

de\end{array} \)
(display: 362,880; 6 \( ; 604,800 \))

# 22 9 books: 4 blue, 3 green, 2 red

a) # of arrangements of 9 books, no restrictions = 
\[ 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 9! = \underline{362,880} \]

b) # of arrangements w/ books grouped by color

6 possible arrangements of color: BGR, BRG, GBR, GRB, RGB, RGG

24 arrangements of 4 blue books: \( 4 \times 3 \times 2 \times 1 = 24 \)

6 arrangements of 3 green books: \( 3 \times 2 \times 1 = 6 \)

2 arrangements of 2 red books: \( 2 \times 1 = 2 \)

Total: \( 6 \times 24 \times 6 \times 2 = \underline{1728} \)

c) Distinguishable arrangements if books of one color are indistinguishable:

\[ \frac{9!}{4! \cdot 3! \cdot 2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4! \cdot 3! \cdot 2!} \]

\[ = \frac{9 \times 8 \times 7 \times 5}{3 \times 2 \times 1} = \underline{1260} \]

\( d) \) How many ways to select 3 books, one of each color, if order of selection does not matter: \( 4 \times 3 \times 2 = \underline{24} \)

e) Same as \( d \) but order of selection matters: \( 6 \times 4 \times 3 \times 2 = \underline{144} \)
38. 35 members selecting 4 officers. \(35 \text{P}_4 = 1,256,640\)

or \(35 \times 34 \times 33 \times 32 = 1,256,640\)

\(P_{\text{VP} T S}\)

43. 7 digit telephone number; 1st digit cannot be zero
   a) only odd digits 1, 3, 5, 7, 9 \(\Rightarrow 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^7 = 7,812,5\)
   b) must end in zero: \(9 \times 10 \times 10 \times 10 \times 10 \times 1 \times 1 = 9 \times 10^5 = 900,000\)
   c) must be multiple of 100: \(9 \times 10 \times 10 \times 10 \times 1 \times 1 \times 1 = 9 \times 10^4 = 90,000\)
   d) 1st 3 digits are 481: \(1 \times 1 \times 1 \times 10 \times 10 \times 10 \times 10 = 10^4 = 10,000\)
   e) no repetition: \(9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 154,320\)

Hint: To find the number of arrangements or permutations possible
for n objects we can use a tree diagram \&/or the
multiplication principle \&/or \(P(n, r)\) if the n objects are
taken r at a time \& there is no repetition

| Multiplication Principle: total number of outcomes successive selections |
| Permutations: \# of different arrangements no repetition |