8.3 Probability Applications of Counting Principles

I. Introduction

Many probability problems involving dependent events can be solved using combinations. Combinations are especially helpful when the number of outcomes is large and a tree diagram is impractical.

II. Review

A. The Basic Probability Principle (p. 369)

Let $S$ be a sample space with $n$ equally likely outcomes. Let event $E$ contain a subset of $m$ of those outcomes. Then the probability that event $E$ occurs, written $P(E)$, is

$$P(E) = \frac{m}{n}$$

ex: A single fair die is rolled. What is the probability that the die shows a number greater than four?

$S = \{1, 2, 3, 4, 5, 6\} \quad n = 6$

$E = \{5, 6\} \quad m = 2 \quad \therefore P(E) = \frac{2}{6} = \frac{1}{3}$

B. Multiplication Principle (p. 432)

If an outcome consists of $n$ successive selections where there are $m_1$ choices for the first selection, $m_2$ choices for the second selection, ..., and $m_n$ choices for the $n^{th}$ selection, the total number of outcomes possible is the product:

$$m_1 \cdot m_2 \cdot \ldots \cdot m_n$$

ex: 3-letter sequences:

(repetition ok) $26 \cdot 26 \cdot 26 = 17,576$

if repetition not ok: $26 \cdot 25 \cdot 24 = 15,600$
C. Complement Rule (p. 37B)

\[ P(E) = 1 - P(E') \quad \text{AND} \quad P(E') = 1 - P(E) \]

EX: If a fair die is rolled, what is the probability that any number but 5 will come up?

Let \( E \) = event that 5 comes up \( \therefore E' \) = event that any number but 5 comes up

\[ P(E) = \frac{1}{6} \quad \therefore P(E') = 1 - \frac{1}{6} = \frac{5}{6} \]

D. Multiple Branches

If there are multiple branches or ways to get an outcome, the total probability of that outcome is the sum of the probabilities of the relevant branches.

EX: \[ \frac{3}{4} \quad \text{A} \quad \frac{2}{3} \quad \text{FM} \quad \frac{1}{4} \quad \text{B} \quad \frac{2}{3} \quad \frac{1}{4} \quad \frac{1}{5} \]

\[ P(\text{A} \cap S) = P(A) \cdot P(S|A) = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2} \]

\[ P(\text{A} \cap \text{Su}) = P(A) \cdot P(\text{Su}|A) = \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6} \]

\[ P(\text{B} \cap S) = P(B) \cdot P(S|B) = \frac{2}{3} \cdot \frac{2}{5} = \frac{2}{15} \]

\[ P(\text{B} \cap \text{Su}) = P(B) \cdot P(\text{Su}|B) = \frac{2}{3} \cdot \frac{3}{5} = \frac{1}{5} \]

\[ P(S) = P(\text{A} \cap S) + P(\text{B} \cap S) = \frac{1}{2} + \frac{2}{15} = \frac{15}{30} + \frac{4}{30} = \frac{19}{30} \]

E. Permutations vs Combinations (p. 446)

Permutations: number of arrangements of \( n \) objects \( r \) at a time

\[ P(n, r) = \frac{n!}{(n-r)!} \quad \text{without repetition, when order matters} \]

Combinations: number of subsets of \( n \) objects \( r \) at a time

\[ ^n\text{C}_r = \frac{n!}{(n-r)! \cdot r!} \quad \text{without repetition when order does not matter} \]
ex: # 4  A basket contains 7 red and 4 yellow apples. A sample of 3 is drawn
Find the probability that the sample contains more red
than yellow apples.

\[ \text{More red than yellow} = 2R \text{ and } 1Y \quad \text{OR} \quad 3R \text{ and } 0Y \]

\[
P(2R \text{ and } 1Y \text{ OR } 3R \text{ and } 0Y) = \frac{\binom{7}{2} \times \binom{4}{1}}{\binom{11}{3}} + \frac{\binom{7}{3} \times \binom{4}{0}}{\binom{11}{3}}
\]

\[
= \frac{21 \times 4 + 35 \times 1}{165} = \frac{84 + 35}{165} = \frac{119}{165} \approx 0.721
\]

ex: # 12  Two cards are drawn at random from a deck of 52.
Find the probability of getting No card higher than 8.
Cards which are not higher than 8: A, 2, 3, 4, 5, 6, 7, 8
or \( \bullet, \spadesuit, \heartsuit, \diamondsuit \)

\[
P(\text{No card higher than 8}) = \frac{32 \times 2}{52 \times 2} = \frac{496}{1008} = \frac{348}{768} \approx 0.454
\]

ex: # 16  26 slips of paper, each with a different letter
One slip drawn; letter recorded (in order) paper replaced
This is done 5 times.

\[
P(\text{word contains no "X", "Y", or "Z")} = \frac{23 \times 23 \times 23 \times 23 \times 23}{26 \times 26 \times 26 \times 26 \times 26}
\]

\[
= \frac{23^5}{26^5} = \frac{6,436,343}{11,881,376} \approx 0.542
\]

ex: # 41  Find probability of getting a 5-card poker hand with
four of a kind.

There are 13 ways to get 4 of a kind (Four Aces, Four 2, Four 3, ..., Four Q, Four K)
Once we have 4 of a kind, there are 48 cards remaining from which the
fifth card can be drawn.

\[
P(\text{Four of a kind}) = \frac{13 \times 48 \times 1}{52 \times 5} = \frac{13 \times 48}{2,598,960} = \frac{624}{2,598,960} = 0.000241
\]
EX: like #50

Five books are to be selected at random from among
6 cookbooks (C); 9 autobiographies (A); and 5 mysteries (M).
Find the probabilities that the selection consists of:

a) 4 cookbooks & 1 mystery
\[ \frac{6C4 \times 5C1 \times 9C0}{20C5} \]

b) Exactly 3 autobiographies
\[ \frac{9C3 \times 11C2}{20C5} \]

c) 2 cookbooks; 2 autobiographies; 1 mystery
\[ \frac{6C2 \times 9C2 \times 5C1}{20C5} \]

d) At least 3 mysteries
At least 3 means 3 or 4 or 5
\[ \frac{5C3 \times 15C2 + 5C4 \times 15C1 + 5C5}{20C5} \]

e) Exactly 2 non-fiction (A or C)
\[ \frac{15C2 \times 5C3}{20C5} \]

f) No more than 3 cookbooks
\[ \frac{6C0 \times 1 \times 5C0 + 6C1 \times 5C1 + 6C2 \times 5C2 + 6C3 \times 5C3}{20C5} \]

g) More non-fiction than fiction
3 non-fiction AND 2 fiction
\[ \frac{(15C3 \times 5C2)}{20C5} \]
OR 4 non-fiction AND 1 fiction
\[ \frac{(15C4 \times 5C1)}{20C5} \]
OR 5 non-fiction AND 0 fiction
\[ \frac{(15C5 \times 5C0)}{20C5} \]
Experiment 32

S: \{ 2 \text{ defective}, 9 \text{ non-defective} \}

E: \{ 0 \text{ defective}, 3 \text{ non-defective} \}

There are \( \binom{11}{3} = 165 \) ways to choose 3 printers.

There are \( 9 \times 2 = 18 \) ways to choose 3 non-defective printers.

\[ P(3 \text{ non-defective drawn from eleven}) = \frac{84}{165} = \frac{28}{55} \approx 0.509 \]

III. Birthdays

The probability that at least 2 of \( n \) people will have the same birthday is:

\[ 1 - \frac{P(365, n)}{(365)^n} \]

Example 21

\( n = 435 \)

Since \( n > 365 \), it is a certain event that at least 2 of the 435 reps will have the same birthday.

\[ P(\text{at least 2 reps have same birthday}) = 1 \]