8.4 Binomial Probability

I. Bernoulli Trials
   A. Introduction

   Many probability problems are concerned with experiments in which an event is repeated many times. For example: finding the probability of getting 3 tails in 5 tosses of a coin, or finding the probability of getting a hit 2 out of 3 times at bat. Probability problems of this kind are called Bernoulli trials problems, or Bernoulli processes. They are named after the Swiss mathematician Jakob Bernoulli (1654-1705). In each case, one outcome is designated a success, and any other outcome is considered a failure.

   Note: If the probability of a success in a single trial is $p$, the probability of failure will be $1 - p$.

   Ex: $P(\text{success}) = .3$ ; $P(\text{failure}) = 1 - .3 = .7$

B. Conditions for Bernoulli Trial Problems (Binomial Experiments)

   A Bernoulli trial problem or binomial experiment, must satisfy the following conditions:

   1. The same experiment is repeated a fixed number of times.
   2. There are only two possible outcomes, success and failure.
   3. The repeated trials are independent, so that the probability of success remains the same for each trial.
II. Binomial Probability

If \( p \) is the probability of success in a single trial of a binomial experiment, the probability of \( x \) successes and \( n-x \) failures in \( n \) independent repeated trials of the experiment is:

\[
\binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}
\]

Note: \( \binom{n}{x} \) or \( nC_x \) give the number of ways the desired outcome can occur. The probability of each of these is given by \( p^x \cdot (1-p)^{n-x} \) where \( p^x \) is the probability of \( x \) successes and \( (1-p)^{n-x} \) is the probability of \( n-x \) failures.

Recall:
- The multiplication principle p. 432
  There are \( m_1 \cdot m_2 \cdot m_3 \cdot \ldots \cdot m_n \) different ways to make a sequence of \( n \) choices where \( m_i \) is the number of ways to make the \( i \)th choice.
- The Union Rule for Mutually Exclusive Events p. 377
  \( P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) \)
- The Complement Rule p. 378
  \( P(E) = 1 - P(E') \) and \( P(E') = 1 - P(E) \)

III. Pascal's Triangle

Pascal's Triangle is a triangular array of numbers arranged in such a way that the \( x+1 \) entry in the \( n \)th row equals \( \binom{n}{x} \)

\[
\begin{array}{cccccccc}
\text{1st row:} & 1 \\
\text{2nd row:} & 1 & 1 \\
\text{3rd row:} & 1 & 2 & 1 \\
\text{4th row:} & 1 & 3 & 3 & 1 \\
\text{5th row:} & 1 & 4 & 6 & 4 & 1 \\
\text{6th row:} & 1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]
Pascal's Triangle is named in honor of the 17th Century mathematician Blaise Pascal (1623-1662) who was one of the first to use it extensively. However, the triangle was known long before Pascal's time and appears in Chinese and Islamic manuscripts from the eleventh century.

Example: Find $5C3$ using Pascal's Triangle.

Go the $3+1=4$th entry in the $5$th row.

$5C3 = \boxed{10}$

check: on calculator 5 MATH 4 3 3 enter $\rightarrow 10$

Example problems:

#6 p. 473 Family has 5 children; probability of having a girl is $\frac{1}{2}$. Find the probability that the family has at least 3 boys.

Since $P(G) = \frac{1}{2}$, $P(G') = P(B) = 1 - \frac{1}{2} = \frac{1}{2}$

"At least 3 boys" means 3 boys or 4 boys or 5 boys

$P(3B) = 5C3 \times \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^2 = 10 \times \frac{1}{8} \times \frac{1}{4} = \frac{10}{32}$

$P(4B) = 5C4 \times \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^1 = 5 \times \frac{1}{16} \times \frac{1}{2} = \frac{5}{32}$

$P(5B) = 5C5 \times \left(\frac{1}{2}\right)^5 \times \left(\frac{1}{2}\right)^0 = 1 \times \frac{1}{32} \times 1 = \frac{1}{32}$

$P(\text{at least 3 B}) = \frac{10}{32} + \frac{5}{32} + \frac{1}{32} = \frac{16}{32} = \frac{1}{2}$

Note: Only use decimal form of probabilities if they are terminating decimals. Otherwise, use reduced fractions to avoid rounding error.
Example: # 13 A die is rolled 6 times.

Find the probability of rolling no more than 2 fives.

Success: rolling a 5 \( p(\text{success}) = \frac{1}{6} \)

Failure: rolling a 1, 2, 3, 4, or 6 \( p(\text{failure}) = \frac{5}{6} \)

"No more than 2 fives": 0 fives or 1 five or 2 fives

\[
P(\text{zero fives}) = \binom{6}{0} \times \left(\frac{1}{6}\right)^0 \times \left(\frac{5}{6}\right)^6 = 1 \times 1 \times \frac{15625}{46,656} = \frac{15625}{46,656} \]

\[
P(\text{one five}) = \binom{6}{1} \times \left(\frac{1}{6}\right)^1 \times \left(\frac{5}{6}\right)^5 = 6 \times \frac{1}{6} \times \frac{3125}{7776} = \frac{3125}{7776} \]

\[
P(\text{two fives}) = \binom{6}{2} \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^4 = 15 \times \frac{1}{36} \times \frac{625}{1296} = \frac{9375}{46,656} \]

\[
P(\text{no more than two fives}) = \frac{15625}{46,656} + \frac{3125}{7776} + \frac{9375}{46,656} = \frac{21,875}{46,656} \approx 0.477 \]

Example: # 50 p. 475

Flu vaccine prevents a person from getting the flu = success

\( P(\text{success}) = 0.8 \quad P(\text{failure}) = 0.2 \quad N = 83 \)

a) \( P(\text{10 people who were inoculated get the flu}) \)

\[
= P(\text{10 failures}) = \binom{83}{10} \times (0.2)^{10} \times (0.8)^3 \approx 0.0210 \]

b) \( P(\text{no more than 4 of the inoculated people get the flu}) \)

\[
= P(\text{no more than 4 failures}) = P(\text{zero failures}) + P(\text{one failure}) + P(\text{two failures}) + P(\text{three failures}) + P(\text{four failures})
\]

\[
= 83\binom{0}{0} \times (0.2)^0 \times (0.8)^{83} + 83\binom{1}{1} \times (0.2)^1 \times (0.8)^{82} +
83\binom{2}{2} \times (0.2)^2 \times (0.8)^{81} + 83\binom{3}{3} \times (0.2)^3 \times (0.8)^{80} + 83\binom{4}{4} \times (0.2)^4 \times (0.8)^{79}
\]

\[
= 8,004 \times 10^{-5} \]

C) \( P(\text{zero inoculated people get the flu}) \)

\[
= 83\binom{0}{0} \times (0.2)^0 \times (0.8)^{83} \approx 9.046 \times 10^{-9} \]
Additional Examples

#1 \( p \) \( 472 \) \( G = \text{girl} \quad B = \text{boy} \)

Find \( P(\text{exactly 2G + 3B}) \)

\[
P(2G) = \frac{1}{2} \quad \therefore \quad P(2G) = P(B) = 1 - \frac{1}{2} = \frac{1}{2} \quad n = 5 \quad x = 2
\]

\[
\binom{n}{x} \cdot p^x \cdot (1-p)^{n-x} = \binom{5}{2} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(1-\frac{1}{2}\right)^5 = \binom{5}{2} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^3 = 10 \cdot 0.25 \cdot 0.125 = \frac{3125}{10000}
\]

\[= 0.3125, \text{ 3rd entry} \]

#5 \( n = 5 \)

\( p \) \( 473 \) "at least 4 girls" means either 4 or 5 girls

\[
P(\text{at least 4 girls}) = P(4 \text{ girls}) + P(5 \text{ girls}) \quad 5^{\text{th}} \text{ row}, 6^{\text{th}} \text{ entry}
\]

\[
= \binom{5}{4} \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^1 + \binom{5}{5} \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^0
\]

\[= 5 \cdot 0.0625 \cdot 0.5 + 1 \cdot 0.03125 \cdot 1 = 0.15625 + 0.03125 = 0.1875
\]

#13 \( n = 12 \)

\( p \) \( 473 \) "No more than 3 ones" means 0, 1, 2, or 3 ones

\[
P(\text{no more than 3 ones}) = P(\text{zero one}) + P(\text{one one}) + P(\text{two ones}) + P(\text{three ones})
\]

\[
= \binom{12}{0} \cdot \left(\frac{5}{6}\right)^0 \cdot \left(\frac{1}{6}\right)^{12} + \binom{12}{1} \cdot \left(\frac{5}{6}\right)^1 \cdot \left(\frac{1}{6}\right)^{11} + \binom{12}{2} \cdot \left(\frac{5}{6}\right)^2 \cdot \left(\frac{1}{6}\right)^{10} + \binom{12}{3} \cdot \left(\frac{5}{6}\right)^3 \cdot \left(\frac{1}{6}\right)^9
\]

\[= 1 \cdot \left(0.125767698\right) + 12 \cdot \left(0.116666667\right) + 110 \cdot \left(0.13458778\right) + 792 \cdot \left(0.13458798\right)
\]

\[= 0.125767698 + 0.13458778 + 0.13458778 + 0.13458798 = 0.877829887 \approx 0.875
\]
Ex: The probability that a randomly selected loan recipient qualifies for the regular rate is .9. Find the probability that at least 9 out of 10 recipients qualify.

\[ P(\text{Quality for regular rate}) = .9 \quad P(\text{not qualified}) = 1 - .9 = .1 \]

\[ n = 10 \quad P(X \geq 9) = P(9) + P(10) = \]
\[ = \binom{9}{9} \cdot .9^9 \cdot .1^1 + \binom{10}{10} \cdot .9^{10} \cdot .1^0 \\
= 10 \cdot (387420499) \cdot .1 + (1) \cdot (3486784401) \cdot (1) \\
= .387420499 + .3486784401 = .7360989391 \approx .736 \]

# 48 p.475

"at most 2 sets of twins" means 0, 1, or 2

\[ P(\text{twins}) = .012 \quad P(\text{not twins}) = 1 - .012 = .988 \quad n = 100 \]

\[ P(\text{at most 2 set of twins}) = P(0) + P(1) + P(2) = \]
\[ = \binom{100}{0} \cdot (.012)^0 \cdot (.988)^{100} + \binom{100}{1} \cdot (.012)^1 \cdot (.988)^{99} + \binom{100}{2} \cdot (.012)^2 \cdot (.988)^98 \\
= 1(1)(.2990160215) + 100(.012)(.302647795) + 4950(.000144)(.3063234792) \\
= .2990160215 + .302647795 + .00081885 = .880548942 \approx .881 \]

\[ n = 10 \quad \text{Find } P(7 \text{ out of 10 questions correct}) \]

Like #33 \[ P(\text{correct}) = \frac{1}{5} = .2 \quad P(\text{incorrect}) = 1 - .2 = .8 \]

\[ P(7 \text{ correct}) = \binom{10}{7} \cdot (.2)^7 \cdot (.8)^3 = 120(.0000128)(.512) \]
\[ = .006786432 \approx .00072 \]