1.1 Limits: A Numerical and Graphical Approach

I. Limits

A. Informal Definition of a Limit

Jessica has a new boyfriend. His name is Ben. Jessica's father is not fond of Ben. He imposes a midnight curfew on Jessica and warns her that each time she is late, he will cut her weekly allowance in half. Currently Jessica's weekly allowance is $100.

According to the rule, the first time Jessica is late her allowance will drop from $100 to $50. The second time she is late, her allowance will drop from $50 to $25. The third time she is late, her allowance will drop from $25 to $12.50, and so on.

According to this model, as the number of times she is late increases, her allowance will get closer and closer to zero. Theoretically (assuming we can make a coin for any denomination, no matter how small), it will never actually reach zero. In mathematical terms, we can say that the *limit* of her allowance is zero.

The *limit* of a function is an output value (L) the function gets closer and closer to as the input value gets closer and closer to some number (a).

B. Limit Notation

An arrow, $\rightarrow$, stands for the word "approaches".
Thus the statement, "as x approaches 3, y approaches 7" can be written "as $x \rightarrow 3$, $y \rightarrow 7$".

$\lim_{x\to 3}(2x+1)=7$ is read as "the limit, as x approaches 3, of 2x + 1, is 7."

As $x \rightarrow 3^-$ is read "as x approaches 3 from the left". (i.e., x < 3; for example: x = 2.9, x = 2.99)
As $x \rightarrow 3^+$ is read "as x approaches 3 from the right". (i.e., x > 3; for example: x = 3.1, x = 3.01))

$\lim_{x\to a^-} f(x)=L$ is read as "the limit, as x approaches a from the left, of f of x is L". It is called the **left-hand limit**.

$\lim_{x\to a^+} f(x)=L$ is read as "the limit, as x approaches a from the right, of f of x is L". It is called the **right-hand limit**.

C. Formal Definition of a Limit

As x approaches a, the limit of f(x) is L, written $\lim_{x\to a} f(x)=L$, if all values of f(x) are close to L for values of x that are sufficiently close, but not equal, to a. The limit L must be a unique real number.

D. Theorem on One-Sided Limits

As x approaches a, the limit of f(x) is L if the left- hand and right-hand limits both exist and both are L. That is,

if $\lim_{x\to a^-} f(x)=L$ and $\lim_{x\to a^+} f(x)=L$, then $\lim_{x\to a} f(x)=L$.

Note: The converse of this theorem is also true. If the overall limit is L, then the left- hand and right-hand limits both exist and both are L.

II. Finding Limits

A. A Numerical Approach: The Table Method

To find $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$, and / or $\lim_{x \to a} f(x)$:

1. Enter the equation into your calculator at y_1 = and adjust the window so you can see the key characteristics of the graph.

2. If you have not done so already, go to the TBLSET window under 2^{nd} WINDOW and set Indpnt on Ask. Leave Depend on Auto. Go to the TABLE at 2^{nd} GRAPH.

3. To find $\lim_{x \to a^-} f(x)$, enter in the table three x-values less than a (i.e., a – .1, a – .01, a – .001). Observe what happens to the y-values as the x-values get closer to a. If they seem to be getting closer to a specific y-value, record that number as the left limit of f(x).

   To find $\lim_{x \to a^+} f(x)$, enter in the table three x-values greater than a (i.e., a + .1, a + .01, a + .001). Observe what happens to the y-values as the x-values get closer to a. If they seem to be getting closer to a specific y-value, record that number as the right limit of f(x).

   If $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$, then $\lim_{x \to a} f(x) = L$.

   If $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ are not the same, then $\lim_{x \to a} f(x) = \text{DNE}$.

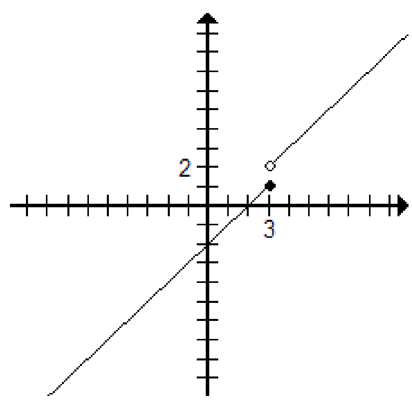
Example 1 Consider the function given by $f(x) = \begin{cases} x-2, & \text{for } x \le 3 \\ x-1, & \text{for } x > 3 \end{cases}$. Find the following limits.

When necessary, state that the limit does not exist.

a. Find $\lim_{x \to 3^-} f(x)$.

As x approaches 3 from the left, y is approaching 1.
Thus $\lim_{x \to 3^-} f(x) = 1$.

$x \to 3^-$	y
2.9	.9
2.99	.99
2.999	.999



b. Find $\lim_{x \to 3^+} f(x)$.

As x approaches 3 from the right, y is approaching 2.
Thus $\lim_{x \to 3^+} f(x) = 2$.

$x \to 3^+$	y
3.1	2.1
3.01	2.01
3.001	2.001

c. Find $\lim_{x \to 3} f(x)$.

Since the left and right limits are not the same, $\lim_{x \to 3} f(x)$ does not exist (DNE).

B. A Graphical Approach: The Wall Method

To find $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$, and / or $\lim_{x \to a} f(x)$:

1. If you are given a graph, skip to step 2.

   If you are not given a graph, enter the equation into your calculator at y_1 = and adjust the window so you can see the key characteristics of the graph.

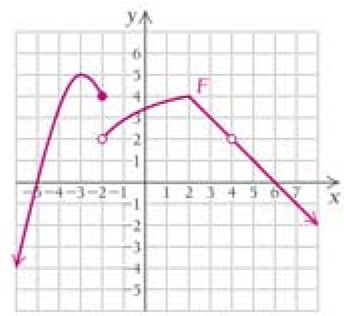
Hint: To enter the equation of a piecewise function such as $f(x)=\begin{cases}2x+4, & x\le 1\\ 7-x, & x>1\end{cases}$,

at y_1 = enter $(2x + 4)(x\le 1) + (7 – x)(x > 1)$.

Inequality symbols such as $\le$ and > are under the TEST menu at 2^{nd} MATH.

2. Draw a vertical line through x = a. This line is called the wall.
To find $\lim\limits_{x\to a^-} f(x)$, on the left side of x = a, follow the curve from left to right until you hit the wall. Mark the location with an x.
To find $\lim\limits_{x\to a^+} f(x)$, on the right side of x = a, follow the curve from right to left until you hit the wall. Mark the location with an x.
If the two exes overlap, $\lim\limits_{x\to a} f(x)$ is the y-value of the point in the center of the x.
If the two exes do not overlap, the overall limit does not exist ($\lim\limits_{x\to a} f(x) = \text{DNE}$).

Example 2 Use the adjacent graph to find each limit. When necessary, state that the limit does not exist.



a. $\lim\limits_{x\to -3} F(x)$

First draw a vertical line at x = -3. Following the curve from left to right, we hit the wall at (-3, 5). Thus $\lim\limits_{x\to -3^-} F(x) = 5$. Following the curve from right to left, we hit the wall at (-3, 5). Thus $\lim\limits_{x\to -3^+} F(x) = 5$.

Since the left and right limits are both 5, $\lim\limits_{x\to -3} F(x) = 5$.

b. $\lim\limits_{x\to -2^-} F(x)$

First draw a vertical line at x = -2. Following the curve from left to right, we hit the wall at (-2, 4). Thus $\lim\limits_{x\to -2^-} F(x) = 4$.

c. $\lim\limits_{x\to -2^+} F(x)$

Following the curve from right to left, we hit the wall at (-2, 2). Thus $\lim\limits_{x\to -2} F(x) = 2$.

d. $\lim\limits_{x\to -2} F(x)$

Since the left and right limits are not the same, $\lim\limits_{x\to -2} F(x)$ does not exist (DNE).

e. $\lim\limits_{x\to 4} F(x)$

First draw a vertical line at x = 4. Following the curve from left to right, we hit the wall at (4, 2). Following the curve from right to left, we again hit the wall at (4, 2). Thus $\lim\limits_{x\to 4^-} F(x) = 2$; $\lim\limits_{x\to 4^+} F(x) = 2$; and $\lim\limits_{x\to 4} F(x) = 2$.

C. <u>Observations</u>

Sometimes $\lim\limits_{x\to a} f(x) = f(a)$, but not always. See example 1 and text example 3 part a.

It is possible for $\lim\limits_{x\to a} f(x)$ to exist even when f(a) does not exist. See example 2 part e and text example 1.

III. Limits Involving Infinity

A. Infinite Limits

When considering $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$, it may happen that f(x) increases or decreases without bound (that is, becomes infinitely large or infinitely small) as x approaches a. In such instances, the limit does not exist in the usual sense. However, we often describe this situation by writing $\lim_{x \to a^-} f(x) = \infty$ or $\lim_{x \to a^-} f(x) = -\infty$.

If as x approaches a from the left, the y-values decrease without bound, we say $\lim_{x \to a^-} f(x) = -\infty$.

If as x approaches a from the left, the y-values increase without bound, we say $\lim_{x \to a^-} f(x) = \infty$.

If as x approaches a from the right, the y-values decrease without bound, we say $\lim_{x \to a^+} f(x) = -\infty$.

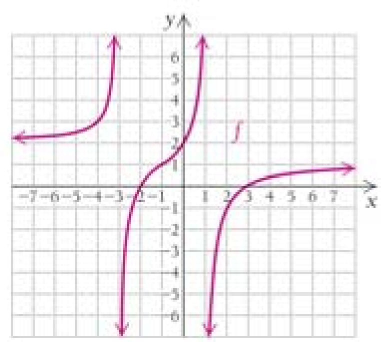
If as x approaches a from the right, the y-values increase without bound, we say $\lim_{x \to a^+} f(x) = \infty$.

B. Limits at Infinity

Sometimes we are concerned with the behavior of a function f as the magnitude of the input variable increases or decreases without bound (that is, the magnitude becomes infinitely large or infinitely small. Such limits are called **limits at infinity** and are written as $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$. The notation $x \to \infty$ can be read "as x increases without bound" or "as x approaches infinity." Similarly, $x \to -\infty$ can be read "as x decreases without bound" or "as x approaches negative infinity."

These limits are approached from one side only:
from the left if approaching positive infinity OR
from the right if approaching negative infinity.

Example 3 Use the adjacent graph to find the following limits. When necessary, state that the limit does not exist.



a. $\lim_{x \to -3} f(x)$

As x approaches -3 from the left, the y-values increase without bound, so we can say $\lim_{x \to -3^-} f(x) = \infty$.

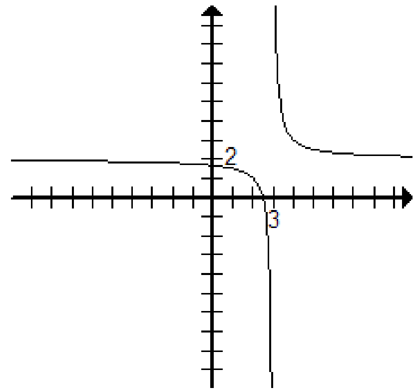
As x approaches -3 from the right, the y-values decrease without bound, so we can say $\lim_{x \to -3^+} f(x) = -\infty$.

Since the left limit and the right limit are not the same, $\lim_{x \to -3} f(x)$ does not exist (DNE).

b. $\lim_{x \to \infty} f(x)$

As x approaches infinity from the left, the y-values get closer and closer to 1. Therefore, $\lim_{x \to \infty} f(x) = 1$.

Example 4 Graph the function $g(x) = \frac{1}{x-3} + 2$, then find the specified limits. When necessary, state that the limit does not exist.



a. $\lim\limits_{x \to \infty} g(x)$

As x approaches infinity from the left, the y-values get closer and closer to 2.
Thus $\lim\limits_{x \to \infty} g(x) = 2$.

b. $\lim\limits_{x \to 3} g(x)$

As x values approach 3 from the left, the y-values decrease without bound. Thus $\lim\limits_{x \to 3^-} g(x) = -\infty$.

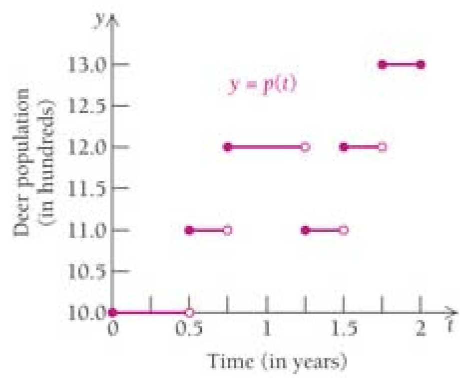
As x values approach 3 from the right, the y-values increase without bound. Thus $\lim\limits_{x \to 3^+} g(x) = \infty$.

Since the left and right limits are not the same, $\lim\limits_{x \to 3} g(x)$ does not exist (DNE).

IV. Applications

Example 5 Population Growth

In a certain habitat, the deer population (in hundreds) as a function of time (in years) is given in the adjacent graph of p.



Use the graph to find the following limits.

a. $\lim\limits_{t \to 1.75^-} p(t)$

As t approaches 1.75 from the left
the deer population is approaching 12 hundred.
Therefore, $\lim\limits_{t \to 1.75^-} p(t) = 1200$.

b. $\lim\limits_{t \to 1.75^+} p(t)$

As t approaches 1.75 from the right
the deer population is approaching 13 hundred.
Therefore, $\lim\limits_{t \to 1.75^+} p(t) = 1300$.

c. $\lim\limits_{t \to 1.75} p(t)$

Since the left and right limits are not the same, $\lim\limits_{t \to 1.75} p(t)$ does not exist (DNE).