1.2 Algebraic Limits and Continuity

I. Algebraic Limits

A. Introduction

In the previous section we learned how to find limits using numerical (Table) and graphical (Wall) methods. In this section we will learn some common limit properties that allow us to calculate limits more efficiently.

B. Limit Properties

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ and c is any constant, then we have the following limit properties.

L1. The limit of a constant is the constant: $\lim_{x\to a} c = c$.

L2. The limit of a power is the power of that limit, and the limit of a root is the root of that limit (assuming n is a positive integer):

$$\lim_{x\to a}\left[f(x)^n\right] = \left[\lim_{x\to a} f(x)\right]^n = L^n \quad \text{and} \quad \lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$$

In the case of the root, we must have L $\geq$ 0 if n is even.

L3. The limit of a sum or a difference is the sum or the difference of the limits:

$$\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$$

L4. The limit of a product is the product of the limits:

$$\lim_{x\to a}\left[f(x)\cdot g(x)\right] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L\cdot M$$

L5. The limit of a quotient is the quotient of the limits:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}, \quad \text{assuming } M \neq 0.$$

L6. The limit of a constant times a function is the constant times the limit of the function:

$$\lim_{x\to a} c\cdot f(x) = c\cdot \lim_{x\to a} f(x) = c\cdot L$$

C. Theorem on Limits of Rational Functions

For any rational function F, with a in the domain of F, $\lim_{x\to a} F(x) = F(a)$.

Note: *Rational functions* include all polynomial functions (i.e., constant functions, linear functions, quadratic functions, cubic functions, etc.) and ratios composed of such functions.

D. Practical Application of the Theorem on Limits of Rational Functions

Unless you are specifically told to find a limit using the numerical (Table) or graphical (Wall) method, try direct substitution first. If you get a real number, you have found the limit.

If you get $\frac{n}{0}$, the limit does not exist. If you get the **indeterminate form** $\frac{0}{0}$, the limit may exist.

Try method a and/or method b.

a. Try factoring the numerator and denominator. Cancel common factors and then try direct substitution again.

b. Try using the Table and/or the Wall method.

Example 1 Use the Theorem on Limits of Rational Functions to find the following limits. When necessary, state that the limit does not exist.

a. $\lim_{x\to -1}\left(3x^5+4x^4-3x+6\right)$

$\lim_{x\to -1}\left(3x^5+4x^4-3x+6\right) = 3(-1)^5+4(-1)^4-3(-1)+6 = -3+4+3+6 = 10$

b. $\lim_{x\to 3}\dfrac{x^2-25}{x^2-5}$

$\lim_{x\to 3}\dfrac{x^2-25}{x^2-5} = \dfrac{(3)^2-25}{(3)^2-5} = \dfrac{9-25}{9-5} = \dfrac{-16}{4} = -4$

Example 2 For the following exercises, the initial substitution of x = *a* yields the form $\frac{0}{0}$. Look for ways to simplify the function algebraically, or use a table and/or graph to determine the limit. When necessary, state that the limit does not exist.

a. $\lim_{x\to -2}\dfrac{x^2-2x-8}{x^2-4}$

$\lim_{x\to -2}\dfrac{x^2-2x-8}{x^2-4} = \lim_{x\to -2}\dfrac{(x-4)(x+2)}{(x-2)(x+2)} = \lim_{x\to -2}\dfrac{x-4}{x-2} = \dfrac{-2-4}{-2-2} = \dfrac{-6}{-4} = \dfrac{3}{2}$

CAUTION: You must write $\lim_{x\to a}$ in front of each expression until you plug in *a* for all exes.

b. $\lim_{x\to 2}\dfrac{x^2+3x-10}{x^2-4x+4}$

$\lim_{x\to 2}\dfrac{x^2+3x-10}{x^2-4x+4} = \lim_{x\to 2}\dfrac{(x+5)(x-2)}{(x-2)(x-2)} = \lim_{x\to 2}\dfrac{x+5}{x-2} = \dfrac{2+5}{2-2} = \dfrac{7}{0}$ Limit does not exist.

Example 3 Use the Limit Properties to find the following limit. If it does not exist, state that fact.

$\lim_{x\to 3}\sqrt{x^2-16}$

$\lim_{x\to 3}\sqrt{x^2-16} = \lim_{x\to 3}\left(x^2-16\right)^{\frac{1}{2}} = \left(\lim_{x\to 3}\left(x^2-16\right)\right)^{\frac{1}{2}} = \left(\lim_{x\to 3}x^2-\lim_{x\to 3}16\right)^{\frac{1}{2}} = \left(\left(\lim_{x\to 3}x\right)^2-16\right)^{\frac{1}{2}} =$

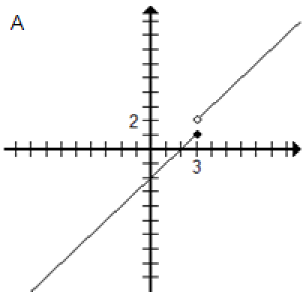
$\left(3^2-16\right)^{\frac{1}{2}} = (9-16)^{\frac{1}{2}} = (-7)^{\frac{1}{2}} = \sqrt{-7}$ Limit does not exist.

II. Continuity

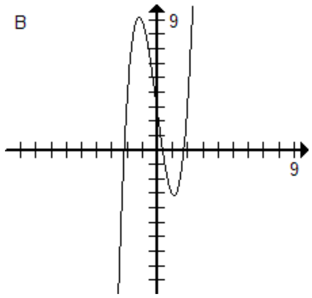
A. Intuitive Definition

A function is **continuous** on an interval of real numbers if its graph on that interval can be traced without lifting the pencil from the paper. If there is any point in the interval where a "jump" or "gap" or "hole" occurs, then the function is not continuous on that interval.

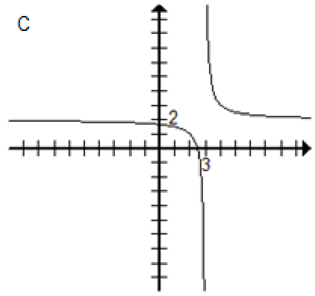
Example 4 Determine whether each of the functions shown is continuous over the interval (-9, 9).



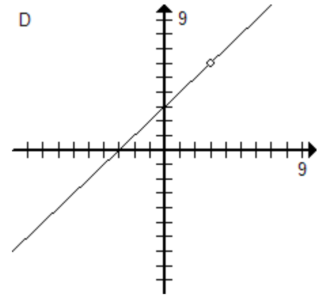
A is discontinuous at x = 3. (jump)



B is continuous.



C is discontinuous at x = 3. (gap)



D is discontinuous at x = 3. (hole)

B. Formal Definition

A function *f* is **continuous** at x = a if:

a. f(a) exists. [The function exists (has an output) at x = a.]

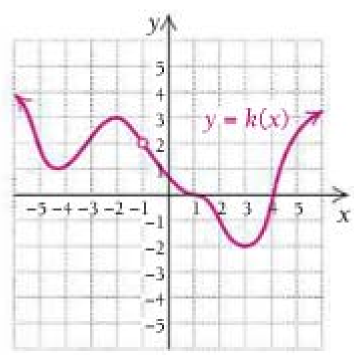
b. $\lim_{x \to a} f(x)$ exists. [The limit as x $\to$ a exists.]

c. $\lim_{x \to a} f(x) = f(a)$ [The limit as x $\to$ a is the same as the output, f(a).]

Note: A function is **continuous over an interval I** if it is continuous at each point in I.

If any part of the continuity definition fails, then the function is discontinuous at x = a.

Example 5 Use the adjacent graph to answer the following questions. If an expression does not exist, state that fact.



a. Find $\lim_{x \to -1} k(x)$. **2**

b. Find k(–1). **DNE**

c. Is k continuous at x = –1? Why or why not? **No** **k(–1) DNE**

d. Find $\lim_{x \to 3} k(x)$. **–2**

e. Find k(3). **–2**

f. Is k continuous at x = 3? Why or why not? **Yes.** $\lim_{x \to 3} k(x) = k(3)$

Example 6 Is the function given by $F(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & \text{for } x \neq 1 \\ 4, & \text{for } x = 1 \end{cases}$ continuous at x = 1? Why or why not?

a. Does F(1) exist? **Yes. F(1) = 4**

b. Does $\lim_{x \to 1} F(x)$ exist? **Yes.**

$$\lim_{x \to 1} F(x) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)(x+1)}{(x-1)} = \lim_{x \to 1}(x+1) = 1 + 1 = 2$$

c. Does $\lim_{x \to 1} F(x) = F(1)$? **No. Therefore F(x) is not continuous at x = 1.**

Example 7 Is the function given by $f(x)=\frac{1}{x^2-6x+8}$ continuous at x = 3?

a. Does f(3) exist? **Yes.** $f(3)=\frac{1}{3^2-6(3)+8}=\frac{1}{9-18+8}=\frac{1}{-1}=-1$

b. Does $\lim_{x\to 3} f(x)$ exist? **Yes.** $\lim_{x\to 3} f(x)=\frac{1}{3^2-6(3)+8}=\frac{1}{9-18+8}=\frac{1}{-1}=-1$

c. Does $\lim_{x\to 3} f(x)$ = f(3)? **Yes. Therefore f(x) is continuous at x = 3.**

Example 8 A lab technician controls the temperature T inside a kiln. From an initial temperature of 0 degrees Celsius ($^\circ$ C), he allows the kiln to increase by 2 $^\circ$ C per minute for the next 60 minutes. After the 60th minute, he allows the kiln to cool at the rate of 3 $^\circ$ C per minute. The temperature function T is defined by

$$T(t)=\begin{cases} 2t, & \text{for } t \le 60, \\ k-3t, & \text{for } t > 60. \end{cases}$$

a. Find k such that T is continuous at t = 60.

T(60) does exist: T(60) = 2(60) = 120.

$\lim_{t\to 60^-} T(t) = 2(60) = 120$ So for T(t) to be continuous at t = 60, the right limit must equal the left limit, so $\lim_{t\to 60^+} T(t)$ must equal 120.

$\lim_{t\to 60^+} k-3t$ must equal 120 implies k – 3(60) = 120 $\Rightarrow$ k – 180 = 120 $\Rightarrow$ k = 300

b. Explain why T must be continuous at t = 60 min.

T must be continuous at t = 60 min because the kiln will cool over a continuous scale.