1.3 Average Rates of Change

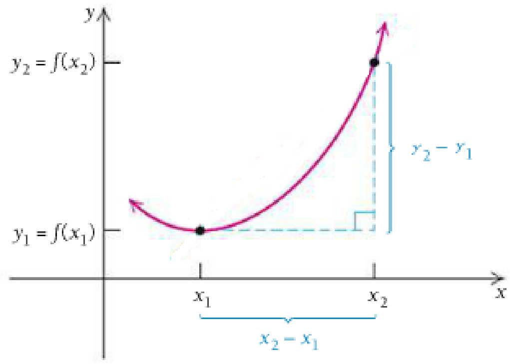
I. Average Rates of Change

A. Introduction

Let's consider a function y = f(x) and two inputs, x_1 and x_2.

$x_2 - x_1$ is **the change in x** or **the change in input**.

$y_2 - y_1$ is **the change in y** or **the change in output** where $y_1 = f(x_1)$ and $y_2 = f(x_2)$.



B. Definition

The **average rate of change of y with respect to x**, as x changes from x_1 to x_2, is the ratio of the change in output to the change in input:

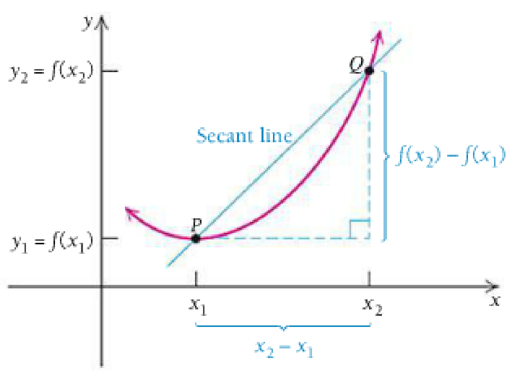
$$\frac{y_2 - y_1}{x_2 - x_1} \quad \text{where } x_2 \neq x_1.$$

In other words, the average rate of change is the slope of the line between the two points (x_1, y_1) and (x_2, y_2).

C. The Slope of the Secant Line

Let's call our first point, (x_1, y_1), P and our second point, (x_2, y_2), Q. The line which passes through P and Q, denoted $\overrightarrow{PQ}$, is called the **secant line**. The **slope of the secant line** (m_{sec}) can be interpreted as the average rate of change of f from x_1 to x_2. And since

$y_1 = f(x_1)$ and $y_2 = f(x_2)$, $\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = m_{sec}$ where $x_2 \neq x_1$.

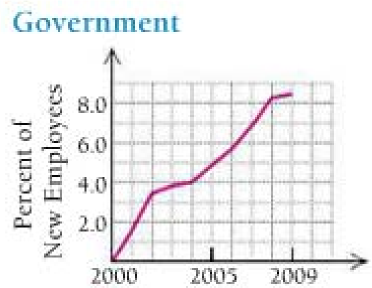


Example 1 **Use the adjacent graph to estimate the average rate of change of the percentage of new employees in government:**

a. **from 2000 to 2005**

(2000, 0) (2005, 4.9)

$\frac{4.9-0}{2005-2000}=\frac{4.9}{5}$ = .98% increase per year



b. **from 2005 to 2009**

(2005, 4.9) (2009, 8.5)

$\frac{8.5-4.9}{2009-2005}=\frac{3.6}{4}$ = .90% increase per year

c. **from 2000 to 2009**

(2000, 0) (2009, 8.5)

$\frac{8.5-0}{2009-2000}=\frac{8.5}{9}\approx$ .94% increase per year

Example 2 **Credit Card Debt**

When a balance of $5,000 is owed on a credit card and interest is being charged at a rate of 14% per year, the total amount owed after *t* years, A(t), is given by

$A(t)=5000(1.14)^t$ **.**

Find $\frac{A(3)-A(2)}{3-2}$**, and interpret this result.**

$$\frac{A(3)-A(2)}{3-2}=\frac{5000(1.14)^3-5000(1.14)^2}{3-2}=\frac{909.72}{1}=909.72$$

$909.72 is the average annual increase in the debt from the second to the third year.

Example 3 **Average Velocity**

In *t* seconds, an object dropped from a certain height will fall s(t) feet, where

$s(t)=16t^2$ **.**

a. **Find s(5) – s(3). What does this represent?**

$s(5) – s(3) = 16(5)^2 – 16(3)^2 = 16(25) – 16(9) = 400 – 144 = 256$ ft

256 ft is the distance traveled between t = 3 sec and t = 5 sec.

b. **What is the average rate of change of distance with respect to time during the period from 3 to 5 second? This is known as the average velocity or speed**.

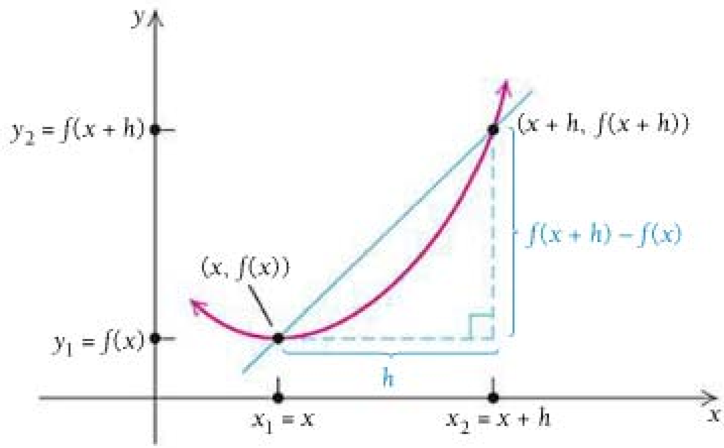
average velocity = $\frac{s(5)-s(3)}{5-3}=\frac{256}{2}=128\,\frac{\text{ft}}{\text{sec}}$

II. Difference Quotients as Average Rates of Change

A. Introduction

There is an alternate notation for average rate of change which does not require subscripts. Instead of x_1, we will simply write x and instead of x_2, we will write x + h. We can think of h as the horizontal distance between the two inputs. Using this new notation, the average rate of change, also called the **difference quotient**, is given by

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h) - f(x)}{h} \quad \text{where } h \neq 0.$$



B. Definition

The average rate of change of f with respect to x is also called the **difference quotient**. It is given by

$$\frac{f(x+h) - f(x)}{h} \quad \text{where } h \neq 0.$$

The difference quotient is equal to the slope of the secant line from the point (x, f(x)) to the point (x + h, f(x + h)).

C. Caution

In general, f(x + h) does NOT equal f(x) + f(h).

D. Note

It is generally preferable to simplify a difference quotient algebraically before evaluating it at specific values of x and h.

Example 5 For each function, (1) find a simplified form of the difference quotient and (2) complete the following table.

x	h	$\frac{f(x+h)-f(x)}{h}$
5	2	
5	1	
5	.1	
5	.01	

a. $f(x)=\frac{2}{x}$

1. $f(x + h) = \frac{2}{x+h}$

$$f(x + h) - f(x) = \frac{2}{x+h}-\frac{2}{x}=\left(\frac{x}{x}\right)\left(\frac{2}{x+h}\right)-\left(\frac{x+h}{x+h}\right)\left(\frac{2}{x}\right)=\frac{2x-2x-2h}{x(x+h)}=\frac{-2h}{x(x+h)}$$

$$\frac{f(x+h)-f(x)}{h}=\frac{\frac{-2h}{x(x+h)}}{h}=\left(\frac{-2h}{x(x+h)}\right)\left(\frac{1}{h}\right)=\frac{-2}{x(x+h)}$$

2.

x	h	$\frac{-2}{x(x+h)}$
5	2	$\frac{-2}{35}$
5	1	$\frac{-1}{15}$
5	.1	$\frac{-4}{51}$
5	.01	$\frac{-40}{501}$

b. $f(x)=x^2-4x$

1. $f(x + h) = (x + h)^2 - 4(x + h) = x^2 + 2xh + h^2 - 4x - 4h$

$$f(x + h) - f(x) = x^2 + 2xh + h^2 - 4x - 4h - (x^2 - 4x) = x^2 + 2xh + h^2 - 4x - 4h - x^2 + 4x$$
$$= 2xh + h^2 - 4h$$

$$\frac{f(x+h)-f(x)}{h}=\frac{2xh+h^2-4h}{h}=\frac{h(2x+h-4)}{h}=2x+h-4$$

2.

x	h	2x + h – 4
5	2	8
5	1	7
5	.1	6.1
5	.01	6.01