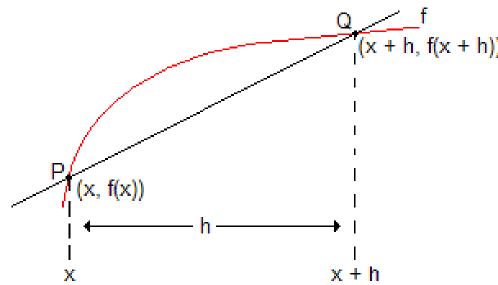
1.4 Differentiation Using Limits of Difference Quotients

I. Rationale

In this section we will define one of the most important concepts of calculus, the derivative. Simply stated, the derivative tells us the slope of a curve or, equivalently, the instantaneous rate of change of a function at a specified input, x. Knowing how to calculate the derivative will enable us to answer questions such as "How fast do the blood cells in a capillary move in the seconds after a certain drug is injected?" or "How does the manufacturing cost per item change as the number of items manufactured changes?" or "What is the velocity of the space shuttle 30 seconds after take-off?" One of the great strengths of calculus is that it allows us to calculate instantaneous rates of change for systems that are in a constant state of flux.

II. Review: Average Rate of Change = Slope of the Secant Line = the Difference Ouotient

As shown in the adjacent graph, P is a point on function f with coordinates (x, f(x)). Q is another point on function f. It is h units to the right of P, so its coordinates are (x+h, f(x+h)).



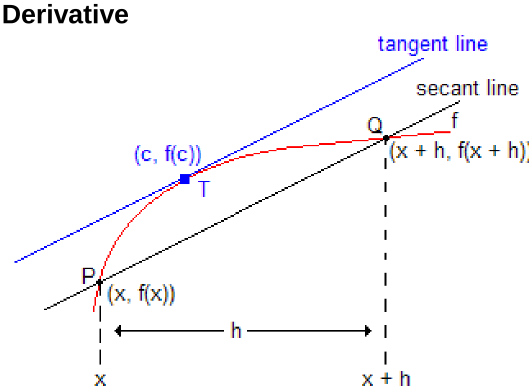
To find the *average rate of change* of the function f, we calculate the slope of the *secant line* which cuts through the function at points P and Q. Secant comes from the Latin *secare* which means "to cut". The slope of the secant line is also called the *difference quotient.*

$$m_{sec} = \frac{f(x+h)-[f(x)]}{h}, \quad \text{where } h \neq 0.$$

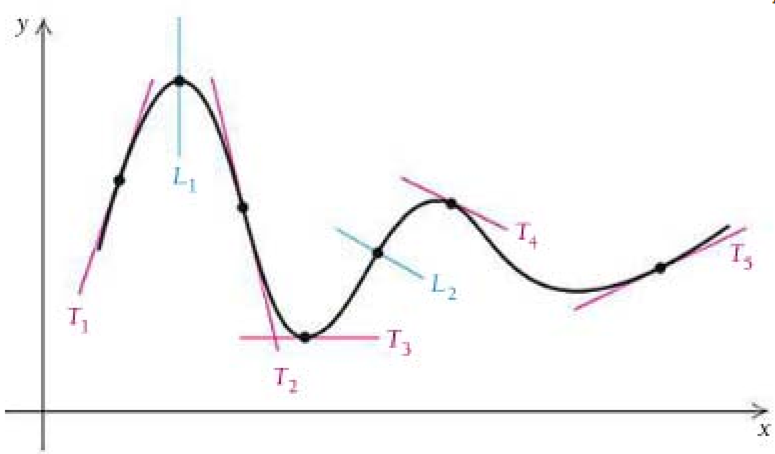
III. Instantaneous Rate of Change = Slope of the Tangent Line = Derivative

A. Tangent Line

As the distance (h) between points P and Q gets smaller and smaller, the two points move closer together until they finally merge into a single point T called the **point of tangency**. The line which touches the curve at point T is called the **tangent line** from the Latin *tangere* which means "to touch".



In the figure below, all the lines are tangent lines except L_1 and L_2.



B. Instantaneous Rate of Change = Slope of Tangent Line = Derivative

To find the **instantaneous rate of change** of function f at point T, we will calculate the slope of the tangent line. The **slope of the tangent line** is the limit of the difference quotient (m_{sec}) as h approaches zero. Another name for the slope of the tangent line is the **derivative**.

$$m_{tan} = \lim_{h \to 0} \frac{f(x+h) - [f(x)]}{h} \quad \text{where } h \neq 0.$$

C. The Definition of the Derivative

The derivative is denoted by $f'(x)$ which is read as "f prime of x".

For a function y = f(x), its **derivative** is the function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - [f(x)]}{h}$$

provided that the limit exists. If $f'(x)$ exists, then we say that f is **differentiable** at x. We sometimes call f' the **derived function**.

The slope of the line tangent to y = f(x) at x = a is the value of the derivative at x = a; that is, the slope of the tangent line at x = a is $f'(a)$.

IV. How to Calculate a Derivative

1. Find f(x + h). **CAUTION:** $f(x + h) \neq f(x) + h$
2. Find f(x + h) – [f(x)].
3. Find $\frac{f(x+h) - [f(x)]}{h}$.
4. Find $\lim_{h \to 0} \frac{f(x+h) - [f(x)]}{h}$.
5. If given a specified value (a), find $f'(a)$.

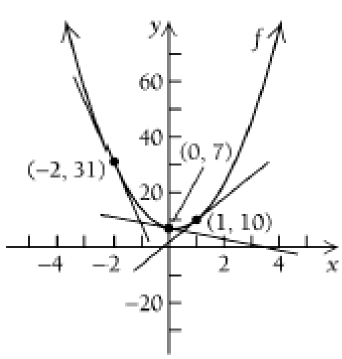
Example 1: Given the function $f(x) = 5x^2 - 2x + 7$, do the following.

a) Graph the function.

b) Draw tangent lines to the graph at points whose x-coordinates are –2, 0, and 1.

c) Find $f'(x)$ by determining $\lim_{h \to 0} \frac{f(x+h) - [f(x)]}{h}$.

d) Find $f'(-2)$, $f'(0)$, and $f'(1)$. These slopes should match those of the lines you drew in part (b).



a, b.

c. 1. $f(x+h) = 5(x+h)^2 - 2(x+h) + 7 = 5(x^2 + 2xh + h^2) - 2x - 2h + 7 =$
$5x^2 + 10xh + 5h^2 - 2x - 2h + 7$

2. $f(x + h) - f(x) = 5x^2 + 10xh + 5h^2 - 2x - 2h + 7 - (5x^2 - 2x + 7) =$
$5x^2 + 10xh + 5h^2 - 2x - 2h + 7 - 5x^2 + 2x - 7 = 10xh + 5h^2 - 2h$

3. $\frac{f(x+h) - f(x)}{h} = \frac{10xh + 5h^2 - 2h}{h} = \frac{h(10x + 5h - 2)}{h} = 10x + 5h - 2$

4. $\lim_{h\to 0} \frac{f(x+h) - [f(x)]}{h} = \lim_{h\to 0}(10x + 5h - 2) = 10x + 5(0) - 2 = 10x - 2.$

d. $f'(-2) = 10(-2) - 2 = -20 - 2 = -22$

$f'(0) = 10(0) - 2 = -2$

$f'(1) = 10(1) - 2\ 10 - 2 = 8$

V. How to Write the Equation of the Tangent Line at a Specified Point

1. Find $f'(x) = \lim_{h\to 0} \frac{f(x+h) - [f(x)]}{h}$.
2. Find m_{tan} by plugging the given x-value into $f'(x)$.
3. Plug m_{tan} and the given x and y-values into the Point-Slope Form, $y - y_1 = m(x - x_1)$, and solve for y.

Example 2: Find an equation of the tangent line to the graph of $f(x) = x^2 - 2x$ at $(-2, 8)$.

$f(x + h) = (x + h)^2 - 2(x + h) = x^2 + 2xh + h^2 - 2x - 2h$

$f(x + h) - [f(x)] = x^2 + 2xh + h^2 - 2x - 2h - [x^2 - 2x] = x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x =$
$2xh + h^2 - 2h$

$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2 - 2h}{h} = \frac{h(2x + h - 2)}{h} = 2x + h - 2$

$\lim_{h\to 0} \frac{f(x+h) - [f(x)]}{h} = \lim_{h\to 0}(2x + h - 2) = 2x + 0 - 2 = 2x - 2 \qquad f'(x) = 2x - 2$

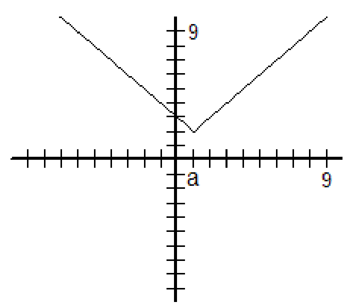
$m_{tan} = 2(-2) - 2 = -4 - 2 = -6$

$y - 8 = -6(x - (-2)) \Rightarrow y - 8 = -6(x + 2) \Rightarrow y - 8 = -6x - 12 \Rightarrow y = -6x - 4$

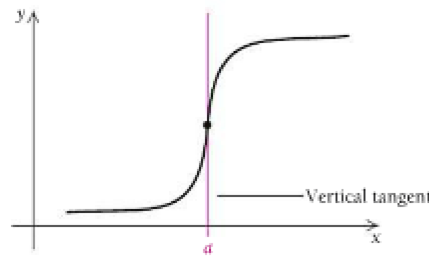
VI. Functions which are not Differentiable

The derivative of a function does not exist at x = a if any of the following three statements are true about the function.

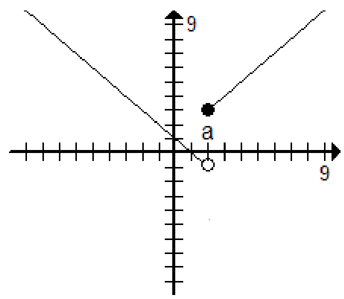
1. The graph of the function has a corner at x = a. Example:



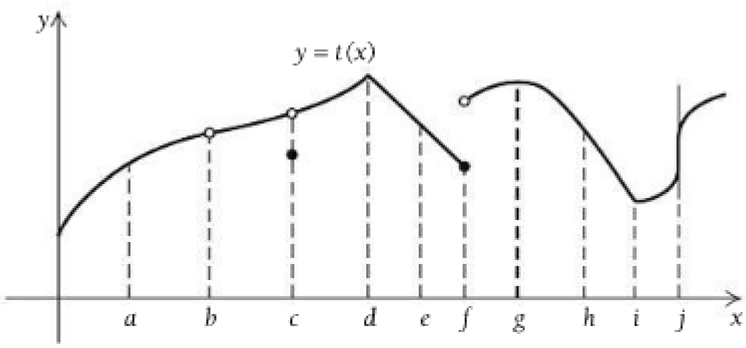
2. The graph of the function has a vertical tangent at x = a. Example:



3. The graph of the function has a discontinuity at x = a. Example:



Example 3: List the points at which each function is not differentiable and state why it is nondifferentiable at that x value.



t(x) is nondifferentiable at x = b, x = c, and x = f because it is discontinuous at those x values.

t(x) is nondifferentiable at x = d and x = i because it has corners at those x values.

t(x) is nondifferentiable at x = j because it has a vertical tangent line at this x values.

VII. Differentiability and Continuity

If y = f(x) has a derivative at x = a, then it is continuous at x = a. In other words, differentiability implies continuity. However, the reverse is not true. Continuity alone is not sufficient to guarantee differentiability