

1.5 Differentiation Techniques: The Power and Sum-Difference Rules

I. The Amazing Power of the Derivative

Why would anyone in the real world ever want to find the derivative of a function?

Historically, the discovery of the process of differentiation led to greater knowledge of the effects of gravity, improvements in pendulum clocks and chronometers, refined methods of navigation, superior lenses, and better telescopes. In modern times, differentiation allows a physician to determine how fast a tumor is growing from its shape. It allows a meteorologist to predict the velocity of a weather balloon 10 seconds after it is released. It helps a production manager determine how a change in the demand for a product will affect its unit price. And it allows a farmer to predict how an aphid infestation will affect his productivity and yield. Do real people in the real world use calculus? Yes, they do, every day!

II. Notation

Four common forms of notation for the derivative of a function are:

1. y' which is read as "y prime"
2. $f'(x)$ which is read "f prime of x"
3. $\frac{dy}{dx}$ which is read "dee y, dee x"
4. $\frac{d}{dx}f(x)$ which is read "dee, dee x of f of x"

When we wish to evaluate a derivative at a number, we write:

$$y'(2) \quad \text{or} \quad f'(3) \quad \text{or} \quad \left. \frac{dy}{dx} \right|_{x=4} \quad [\text{The vertical line is read "evaluated at".}]$$

III. A Few Reminders

A. The Negative Exponent Rule

$$\frac{1}{a^n} = a^{-n} \quad \text{Example: } \frac{1}{x^6} = x^{-6}$$

B. The Rule for Converting a Radical to an Expression with a Fractional Exponent

$$\sqrt[n]{a^m} = a^{\frac{m}{n}} \quad \text{Example: } \sqrt[5]{x^2} = x^{\frac{2}{5}}$$

C. The Meaning of the Derivative

The derivative of a function f is another function f' which gives the slope of the graph of f at any point at which f is differentiable. It also tells the instantaneous rate of change of f at any point at which f is differentiable.

IV. Four Basic Rules of Differentiation

A. The Power Rule

For any real number k , if $y = x^k$, then $\frac{d}{dx}x^k = k \cdot x^{k-1}$.

To take the derivative of x^k , bring the power down in front and subtract one from the exponent.

Hints: The derivative of x is 1. $\frac{d}{dx}x = 1$.

If $f(x)$ contains a power of x in the denominator, rewrite the expression using the negative exponent rule and then take the derivative.

If $f(x)$ contains a radical, convert it to fractional exponent form and then take the derivative.

Example 1: Find $\frac{dy}{dx}$.

a. $y = x^7$ $\frac{dy}{dx} = 7 \cdot x^{7-1} = 7x^6$

b. $y = \frac{1}{x^3}$ $y = \frac{1}{x^3} = x^{-3}$ $\frac{dy}{dx} = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$

c. $y = -\sqrt[4]{x^3}$ $y = -\sqrt[4]{x^3} = -x^{\frac{3}{4}}$ $\frac{dy}{dx} = -\frac{3}{4}x^{\frac{3}{4}-1} = -\frac{3}{4}x^{-\frac{1}{4}} = -\frac{3}{4x^{\frac{1}{4}}} = -\frac{3}{4\sqrt[4]{x}}$

Caution: Be sure that you label your derivative and that you do not indicate that the original function (or the rewritten original function) equals its derivative.

Correct: $y = x^6$ $\frac{dy}{dx} = 6x^5$ **Incorrect:** $y = \frac{1}{x^5} = x^{-5} = -5x^{-5-1} = -5x^{-6} = -\frac{5}{x^6}$

B. The Constant Rule

The derivative of a constant is 0. That is, $\frac{d}{dx}c = 0$.

Hint: The graph of a constant function is a horizontal line and the slope of a horizontal line is 0. Also, the tangent line to a constant function is the function itself.

Example 2: Find $f'(x)$.

$f(x) = 12$ $f'(x) = 0$

C. The Constant Multiple Rule

The derivative of a constant times a function is the constant times the derivative of the function.

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}f(x)$$

Hint: For any constant c , $\frac{d}{dx}c \cdot x = c$.

Caution: Do not confuse the Constant Rule with the Constant Multiple Rule. The derivative of $f(x) + c$ is just $f'(x)$. But the derivative of $c \cdot f(x) = c \cdot f'(x)$. That is, the constant is retained for a product but not for a sum.

Example 3: Find y' .

a. $y = 4x^{-2}$ $y' = 4 \cdot -2x^{-2-1} = -8x^{-3}$

b. $y = \frac{2}{5x^6}$ $y = \frac{2}{5}x^{-6}$ $y' = \frac{2}{5} \cdot -6x^{-6-1} = -\frac{12}{5}x^{-7} = -\frac{12}{5x^7}$

D. The Sum and Difference Rule

1. The derivative of a sum is the sum of the derivatives.

$$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

2. The derivative of a difference is the difference of the derivatives.

$$\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}f(x)-\frac{d}{dx}g(x)$$

Example 4: Find the derivative. $\frac{d}{dx}(5x^2 - 7x + 3)$

$$\frac{d}{dx}f(x) = 5 \cdot 2x^{2-1} - 7 + 0 = 10x - 7$$

V. **Evaluating a Derivative for a Specified x Value**

First find the derivative and then plug in the given x value.

Example 5: If $y = x^3 + 2x - 5$, find $\left. \frac{dy}{dx} \right|_{x=-2}$.

$$\frac{dy}{dx} = 3x^{3-1} + 2 - 0 = 3x^2 + 2 \qquad \left. \frac{dy}{dx} \right|_{x=-2} = 3(-2)^2 + 2 = 3(4) + 2 = 12 + 2 = 14$$

VI. **Writing the Equation of a Tangent Line**

1. Find $f'(x)$.
2. Find m_{tan} by plugging the given x into $f'(x)$.
3. Plug the given x and y, along with m_{tan} into $y - y_1 = m(x - x_1)$ and solve for y.

Example 6: Find an equation (in $y = mx + b$ form) of the tangent line to the graph of $f(x) = x^3 - 2x + 1$ at $(2, 5)$.

1. $f'(x) = 3x^2 - 2$
2. $m_{\text{tan}} = f'(2) = 3(2)^2 - 2 = 3(4) - 2 = 12 - 2 = 10$
3. $y - 5 = 10(x - 2) \rightarrow y - 5 = 10x - 20 \rightarrow y = 10x - 15$

VII. **Finding the Points on the Graph of a Function at which the Tangent Line is Horizontal**

1. Find $f'(x)$.
2. Set $f'(x) = 0$ and solve for x. This gives you the x values of the points.
3. Plug the x values into the original function to find the y values of the points.
4. List the points.

Example 7: Find the points on the graph of $f(x) = -x^3 + x^2 + 5x - 1$ at which the tangent line is horizontal. If none exist, state that fact.

$$1. \quad f'(x) = -3x^2 + 2x + 5$$

$$2. \quad -3x^2 + 2x + 5 = 0 \quad \rightarrow \quad 3x^2 - 2x - 5 = 0 \quad \rightarrow \quad (x + 1)(3x - 5) = 0 \quad \rightarrow$$

$$x + 1 = 0 \quad \rightarrow \quad x = -1 \quad \text{and} \quad 3x - 5 = 0 \quad \rightarrow \quad 3x = 5 \quad \rightarrow \quad x = \frac{5}{3}$$

$$3. \quad f(-1) = -(-1)^3 + (-1)^2 + 5(-1) - 1 = 1 + 1 - 5 - 1 = -4$$

$$f\left(\frac{5}{3}\right) = -\left(\frac{5}{3}\right)^3 + \left(\frac{5}{3}\right)^2 + 5\left(\frac{5}{3}\right) - 1 = -\frac{125}{27} + \frac{25}{9} + \frac{25}{3} - 1 = -\frac{125}{27} + \frac{75}{27} + \frac{225}{27} - \frac{27}{27} = \frac{148}{27}$$

$$4. \quad (-1, -4) \text{ and } \left(\frac{5}{3}, \frac{148}{27}\right)$$

VIII. Applications

“Find the rate of change” means find the derivative.

Example 8: The median weight of a boy whose age is between 0 and 36 months can be approximated by the function

$$w(t) = 8.15 + 1.82t - 0.0596t^2 + 0.000758t^3,$$

where t is measured in months and w is measured in pounds.

Use this approximation to find the following for a boy with median weight:

a. The rate of change of weight with respect to time.

$$w'(t) = 1.82 - 0.1192t + 0.002274t^2$$

b. The weight of the baby at age 10 months.

$$w(10) = 8.15 + 1.82(10) - 0.0596(10)^2 + 0.000758(10)^3 = 21.148 \approx 21 \text{ lbs}$$

c. The rate of change of the baby's weight with respect to time at age 10 months.

$$w'(10) = 1.82 - 0.1192(10) + 0.002274(10)^2 = .8554 \approx .86 \frac{\text{lb}}{\text{month}}$$