1.6 Differentiation Techniques: The Product and Quotient Rules

I. The Product Rule

Let F(x) = f(x) · g(x). Then

$$F'(x) = \frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\left[\frac{d}{dx}g(x)\right] + g(x)\cdot\left[\frac{d}{dx}f(x)\right]$$

The formula is a little clearer if we use prime notation and write the functions as f and g:

$F'(x) = f\cdot g' + g\cdot f'$. [f g-prime plus g f-prime]

The derivative of a product is the first function times the derivative of the second plus the second times the derivative of the first.

Caution: The derivative of a product is <u>not</u> the product of the derivatives. $\frac{d}{dx}(f\cdot g) \neq f'\cdot g'$

<u>Steps for Using the Product Rule</u>

1. Write down f and g.
2. Find f' and g'.
3. Plug into the product rule: $F'(x) = f\cdot g' + g\cdot f'$.
4. Simplify.

Note: Not every product requires the Product Rule. It is often easier to multiply the two functions first and then find the derivative.

Example 1: Differentiate in two ways: first, by using the Product Rule; then by multiplying the expressions before differentiating. Compare your results as a check.

$F(x) = (4x – 3)(2x^2 + 3x + 5)$

Method 1:

1. $f(x) = 4x – 3$ $g(x) = 2x^2 + 3x + 5$
2. $f'(x) = 4$ $g'(x) = 4x + 3$
3. $F'(x) = f\cdot g' + g\cdot f'$ = $(4x – 3)(4x + 3) + (2x^2 + 3x + 5)(4)$
4. $F'(x) = 16x^2 + 12x - 12x - 9 + 8x^2 + 12x + 20 = 24x^2 + 12x + 11$

Method 2:

1. $F(x) = (4x – 3)(2x^2 + 3x + 5) = 8x^3 + 12x^2 + 20x – 6x^2 – 9x – 15 = 8x^3 + 6x^2 + 11x – 15$
2. $F'(x) = 24x^2 + 12x + 11$

II. The Quotient Rule

If $Q(x) = \frac{N(x)}{D(x)}$, then $Q'(x) = \frac{D(x)\cdot N'(x) - N(x)\cdot D'(x)}{[D(x)]^2}$.

Again, the formula is a little clearer if we use prime notation and write the functions as D and N:

$Q'(x) = \frac{D\cdot N' - N\cdot D'}{D^2}$. [D N-prime – N D-prime over D-squared]

The derivative of a quotient is the bottom times the derivative of the top minus the top times the derivative of the bottom over the bottom squared.

If we think of N as high, D as low, N' as dee-high, and D' as dee-low, another way to remember the Quotient Rule is as "low dee-high minus high dee-low over low low".

Caution: The derivative of a quotient is not the quotient of the derivatives. $\frac{d}{dx}\left(\frac{N}{D}\right) \neq \frac{N'}{D'}$.

<u>Steps for Using the Quotient Rule</u>

1. Write down N and D.
2. Find N' and D'.
3. Plug into the quotient rule: $Q'(x) = \frac{D \cdot N' - N \cdot D'}{D^2}$.
4. Simplify.

Note: Not every quotient requires the Quotient Rule. Sometimes it is possible to factor the numerator and/or the denominator first, cancel common factors, and then find the derivative. If you do this, remember that you can only cancel factors involved in multiplication and division. You cannot cancel terms involved in addition and subtraction. Also, you can only cancel a factor if it appears in all terms of the quotient.

Example:

In $\frac{(4x+3)(2)-(2x-1)(4)}{(4x+3)^2}$ we cannot cancel 4x + 3 since it is not a factor in every term.

Hint: $(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$

Example 2: Differentiate two ways: first, by using the Quotient Rule; then, by dividing the expressions before differentiating. Compare your results as a check.

$$Q(x) = \frac{8x^3 - 1}{2x - 1}$$

Method 1:

1. $N(x) = 8x^3 - 1$ $D(x) = 2x - 1$
2. $N'(x) = 24x^2$ $D'(x) = 2$
3. $Q'(x) = \frac{D \cdot N' - N \cdot D'}{D^2} = \frac{(2x-1)(24x^2) - (8x^3-1)(2)}{(2x-1)^2}$
4. $Q'(x) = = \frac{(2x-1)(24x^2) - (8x^3-1)(2)}{(2x-1)^2} = \frac{(2x-1)(24x^2) - (2x-1)(4x^2+2x+1)(2)}{(2x-1)} =$

$$\frac{(2x-1)(24x^2) - (2x-1)(8x^2+4x+2)}{(2x-1)^2} = \frac{(2x-1)[(24x^2) - (8x^2+4x+2)]}{(2x-1)^2} =$$

$$\frac{(16x^2-4x-2)}{(2x-1)} = \frac{(2x-1)(8x+2)}{(2x-1)} = 8x+2 \qquad x \neq \frac{1}{2}$$

Method 2

$Q(x)=\dfrac{8x^3-1}{2x-1} = \dfrac{(2x-1)(4x^2+2x+1)}{(2x-1)} = 4x^2 + 2x + 1$

$Q'(x)= 8x + 2 \quad x \neq \dfrac{1}{2}$

Example 3: **Differentiate the function:** $F(x)=(8x+\sqrt{x})(5x^2+3)$

Method 2:

1. $F(x)=(8x+\sqrt{x})(5x^2+3) = \left(8x+x^{\frac{1}{2}}\right)(5x^2+3) = 40x^3+24x+5x^{\frac{5}{2}}+3x^{\frac{1}{2}}$

2. $F'(x)=120x^2+24+\dfrac{25}{2}x^{\frac{3}{2}}+\dfrac{3}{2}x^{-\frac{1}{2}} = 120x^2+\dfrac{25}{2}x^{\frac{3}{2}}+\dfrac{3}{2}x^{-\frac{1}{2}}+24$

III. Application of the Quotient Rule

A. Total Cost, Revenue, and Profit Functions

Total Cost = variable cost × # of units produced + fixed costs → C(x) = mx + b

Total Revenue = unit price × # of units sold → R(x) = px

Total Profit = Total Revenue – Total Costs → P(x) = R(x) – C(x)

The total cost, total revenue, and total profit functions defined above pertain to the accumulated cost, revenue, and profit when *x* items are produced. Because of economy of scale and other factors, it is common for the cost, revenue (price), and profit for, say, the 10th item to differ from those for the 1000th item. For this reason, a business is often interested in the *average* cost, revenue, and profit associated with the production and sale of *x* items.

B. Average Cost, Revenue, and Profit Functions

If C(x) is the cost of producing *x* items, then the **average cost** of producing *x* items is $\dfrac{C(x)}{x}$.

If R(x) is the revenue from the sale of *x* items, then the **average revenue** from selling *x* items is $\dfrac{R(x)}{x}$.

If P(x) is the profit from the sale of *x* items, then the **average profit** from selling *x* items is $\dfrac{P(x)}{x}$.

Example 4: **Summertime Fabrics finds that the revenue, in dollars, from the sale of *x* jackets is given by $R(x)=85\sqrt{x}$. Find the rate at which the average revenue is changing when 400 jackets have been produced.**

Average Revenue = $A_R = \dfrac{R(x)}{x}=\dfrac{85\sqrt{x}}{x}=\dfrac{85x^{\frac{1}{2}}}{x}=85x^{-\frac{1}{2}}$

$A_R{}'(x) = -\dfrac{85}{2}x^{-\frac{3}{2}}$

$A_R{}'(400)=-\dfrac{85}{2}(400)^{-\frac{3}{2}}=-.00531250$ dollars per jacket

Therefore, when 400 jackets have been produced, average revenue is changing at a rate of – .0053 dollars per jacket.