1.7 The Chain Rule

I. The Extended Power Rule

Suppose that g(x) is a differentiable function of x. Then, for any real number k,

$\frac{d}{dx}[g(x)]^k = k\cdot[g(x)]^{k-1}\cdot\frac{d}{dx}g(x)$. Or written more concisely, $\frac{d}{dx}[g(x)]^k = k\cdot g^{k-1}\cdot g'$

In words, to differentiate a function raised to a power, bring the exponent down in front as a coefficient, subtract 1 from the exponent, and then multiply by the derivative of the inside function.

Caution:

A common error is to forget to multiply by $g'(x)$.
Another common error is to raise the differentiated function to k – 1 instead of the original function.

Example 1: **Differentiate the function:** $y=(7-x)^{55}$**.**

$$y' = 55\cdot(7-x)^{55-1}\cdot -1 = -55(7-x)^{54}$$

Example 2: **Differentiate the function:** $y=\sqrt{1-x}$**.**

$$y=\sqrt{1-x} = (1-x)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(1-x)^{\frac{1}{2}-1}\cdot -1 = -\frac{1}{2}(1-x)^{-\frac{1}{2}} = -\frac{1}{2\sqrt{1-x}}$$

Some problems may require the use of the extended power rule plus other rules such as the sum-difference rule, the product rule, and / or the quotient rule. If possible, simplify before you differentiate.

Example 3: **Differentiate the function:** $F(x)=\left(1+x^3\right)^3-\left(2+x^8\right)^4$**.**

Let $f(x)=\left(1+x^3\right)^3$ and $g(x)=\left(2+x^8\right)^4$. Using the Extended Power Rule separately on each function, we get $f'(x)=3\cdot\left(1+x^3\right)^{3-1}\cdot 3x^2 = 9x^2\cdot\left(1+x^3\right)^2$ and

$g'(x)=4\cdot\left(2+x^8\right)^{4-1}\cdot 8x^7 = 32x^7\cdot\left(2+x^8\right)^3$. We then plug these into the Difference Rule.

$$F'(x)=9x^2\cdot\left(1+x^3\right)^2-32x^7\cdot\left(2+x^8\right)^3$$

Example 4: **Differentiate the function:** $F(x)=(5x+2)^4\cdot(2x-3)^8$**.**

Let's let $f(x)=(5x+2)^4$ and $g(x)=(2x-3)^8$. Then, using the Extended Power Rule we can find $f'(x)$ and $g'(x)$.

$f'(x)=4\cdot(5x+2)^{4-1}\cdot 5=20(5x+2)^3$ and $g'(x)=8\cdot(2x-3)^{8-1}\cdot 2=16(2x-3)^7$

We can now plug into the Product Rule $F'(x)=f\cdot g'+g\cdot f'$ and then simplify.

$$F'(x)=(5x+2)^4\cdot 16(2x-3)^7+(2x-3)^8\cdot 20(5x+2)^3 = (5x+2)^3\cdot(2x-3)^7[16\cdot(5x+2)+20\cdot(2x-3)]$$

$$= (5x+2)^3\cdot(2x-3)^7\cdot[80x+32+40x-60] = (5x+2)^3\cdot(2x-3)^7\cdot(120x-28)$$

Example 5: **Differentiate the function:** $g(x)=\left(\frac{2x+3}{5x-1}\right)^{-4}$ **.**

Let's let the numerator be N = 2x + 3, and the denominator be D = 5x – 1.
Then $N'=2$ and $D'=5$. Now we can use the Quotient Rule to find the derivative of the inside function, $I=\frac{2x+3}{5x-1}$.

$$I'=\frac{D\cdot N'-N\cdot D'}{D^2}=\frac{(5x-1)\cdot 2-(2x+3)\cdot 5}{(5x-1)^2}=\frac{10x-2-10x-15}{(5x-1)^2}=\frac{-17}{(5x-1)^2}$$

Now we can find $g'(x)$ by using the Extended Power Rule.

$$g'(x)=-4\cdot\left(\frac{2x+3}{5x-1}\right)^{-4-1}\cdot\frac{-17}{(5x-1)^2}=68\cdot\left(\frac{2x+3}{5x-1}\right)^{-5}\cdot\frac{1}{(5x-1)^2}=68\cdot\left(\frac{5x-1}{2x+3}\right)^{5}\cdot\frac{1}{(5x-1)^2}$$

$$=68\cdot\frac{(5x-1)^5}{(2x+3)^5}\cdot\frac{1}{(5x-1)^2}=\frac{68\cdot(5x-1)^3}{(2x+3)^5}$$

II. Composition of Functions

In the real world, it is not uncommon for a function's output to depend upon the output of another function. For example, the amount of state income tax a person pays is a function of his adjusted gross income which in turn is a function of his annual earnings. Such functions are called **composite functions**.

In general, the composite function $f\circ g$ (read "f circle g"), the composition of functions f and g, is defined as $(f\circ g)(x)=f(g(x))$. To find $(f\circ g)(x)$, we plug in g(x) for every x in the f function.

Example 6: **For f(x) = x^2 + 3 and g(x) = x – 4, find $(f\circ g)(x)$ and $(g\circ f)(x)$.**

$$(f\circ g)(x)=f(g(x))=f(x-4)=(x-4)^2+3=x^2-8x+16+3=x^2-8x+19$$

$$(g\circ f)(x)=g(f(x))=g(x^2+3)=x^2+3-4=x^2-1$$

Note: In general, $(f\circ g)(x)\neq(g\circ f)(x)$.

III. Decomposing a Composite Function

To decompose a composite function, we think of the given function as a composition of two simpler functions, the inside function [g(x)] and the outside function [f(x)].

Example 7: **Given** $h(x)=\frac{1}{\sqrt{7x+2}}$ **, find f(x) and g(x) such that** $h(x)=(f\circ g)(x)$**. Answers may vary.**

Let 7x + 2 be the inside function, g(x), and $\frac{1}{\sqrt{x}}$ be the outside function, f(x).

$$(f\circ g)(x)=f(g(x))=f(7x+2)=\frac{1}{\sqrt{7x+2}}=h(x)$$

IV. The Chain Rule

The derivative of the composition of $f \circ g$ is given by $\frac{d}{dx}[(f \circ g)(x)] = \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$.

[The derivative of f circle g is equal to f-prime of g times g-prime.]

In words, to find the derivative of $f \circ g$, find the derivative of f(x), replace each x in it with g(x), and then multiply the result by the derivative of g(x).

Hint: f(x) is the outer function and g(x) is the inner function.

Note: The Extended Power Rule is a special case of the Chain Rule.

Example 8: Do this exercise in two ways. First, use the Chain Rule to find the answer. Next, check your answer by finding f(g(x)), taking the derivative and substituting.

$f(u) = \frac{u+1}{u-1}$, $g(x) = u = \sqrt{x}$ Find $(f \circ g)'(4)$.

A. Using the Chain Rule:

First we must find $f'(u)$ and $g'(x)$.

$$f'(u) = \frac{(u-1)\cdot 1 - (u+1)\cdot 1}{(u-1)^2} = \frac{(u-1)-(u+1)}{(u-1)^2} = \frac{u-1-u-1}{(u-1)^2} = \frac{-2}{(u-1)^2}$$

$$g(x) = \sqrt{x} = x^{\frac{1}{2}} \qquad g'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$$

Since $f'(u) = \frac{-2}{(u-1)^2}$ and $g(x) = u = \sqrt{x}$, $f'(g(x)) = \frac{-2}{(\sqrt{x}-1)^2}$

Now we can plug into the Chain Rule

$$\frac{d}{dx}[(f \circ g)(x)] = f'(g(x)) \cdot g'(x) = \frac{-2}{(\sqrt{x}-1)^2} \cdot \frac{1}{2}x^{-\frac{1}{2}} = \frac{-1}{\sqrt{x}(\sqrt{x}-1)^2}$$

Therefore, $(f \circ g)'(4) = \frac{-1}{\sqrt{4}(\sqrt{4}-1)^2} = \frac{-1}{2(2-1)^2} = -\frac{1}{2}$

B. Finding f(g(x)), taking the derivative, and substituting:

$$f(g(x)) = f(\sqrt{x}) = \frac{\sqrt{x}+1}{\sqrt{x}-1} = \frac{x^{\frac{1}{2}}+1}{x^{\frac{1}{2}}-1}$$

By the Quotient Rule,

$$f'(g(x)) = \frac{\left(x^{\frac{1}{2}}-1\right)\cdot\frac{1}{2}x^{-\frac{1}{2}} - \left(x^{\frac{1}{2}}+1\right)\cdot\frac{1}{2}x^{-\frac{1}{2}}}{\left(x^{\frac{1}{2}}-1\right)^2} = \frac{\frac{1}{2}-\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}-\frac{1}{2}x^{-\frac{1}{2}}}{\left(x^{\frac{1}{2}}-1\right)^2} = \frac{-1x^{\frac{1}{-2}}}{\left(x^{\frac{1}{2}}-1\right)^2} = \frac{-1}{x^{\frac{1}{2}}\left(x^{\frac{1}{2}}-1\right)^2} = \frac{-1}{\sqrt{x}(\sqrt{x}-1)^2}$$

Therefore, $(f \circ g)'(4) = \frac{-1}{\sqrt{4}(\sqrt{4}-1)^2} = \frac{-1}{2(2-1)^2} = -\frac{1}{2}$

V. Another Form of the Chain Rule

If y = f(u) and u = g(x), then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Example 9: **Find $\frac{dy}{du}$, $\frac{du}{dx}$, and $\frac{dy}{dx}$ for $y=\sqrt{u}$ and $u=x^2-1$.**

$y=\sqrt{u}=u^{\frac{1}{2}}$ $\quad \frac{dy}{du}=\frac{1}{2}u^{\frac{1}{2}-1}=\frac{1}{2}u^{-\frac{1}{2}}=\frac{1}{2\sqrt{u}}=\frac{1}{2\sqrt{x^2-1}}$

$u=x^2-1$ $\quad \frac{du}{dx}=2x$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{2\sqrt{x^2-1}}\cdot 2x=\frac{x}{\sqrt{x^2-1}}$$

Example 10: **A company is selling laptop computers. It determines that its total profit, in dollars, is given by $P(x) = 0.08x^2 + 80x$ where x is the number of units produced and sold. Suppose that x is a function of time, in months, where $x = 5t + 1$.**

a. Find the total profit as a function of time t.

b. Find the rate of change of total profit when t = 48 months.

a. $(P\circ x)(t)=P(x(t))=P(5t+1)=.08(5t+1)^2+80(5t+1)=.08(25t^2+10t+1)+400t+80$

$=2t^2+.8t+.08+400t+80=2t^2+400.8t+80.08$

$P(t)=2t^2+400.8t+80.08$

b. $P'(t)=4t+400.8$

$P'(48)=4(48)+400.8=\$592.80$ per month