1.8 Higher-Order Derivatives

I. Introduction to Higher-Order Derivatives

The derivative, f', of function f is also a function. As such, it too has a derivative.

The derivative of f' is the **second derivative** of f and is denoted by f''. i.e., $\frac{d}{dx}[f'(x)] = f''(x)$

The derivative of f'' is the **third derivative** of f and is denoted by f'''. i.e., $\frac{d}{dx}[f''(x)] = f'''(x)$

By continuing this process, you obtain **higher-order derivatives** of function f.

The n^{th} derivative of f(x) is the instantaneous rate of change of the (n – 1) derivative with respect to x. For example, f'' is the instantaneous rate of change of f' with respect to x. It measures how the rate of change is itself changing. That is, the second derivative tells whether the rate of change (f') is speeding up or slowing down.

II. Notation for Higher-Order Derivatives

1.	1^{st} derivative	f'	$\frac{dy}{dx}$	y'	$\frac{d}{dx}[f(x)]$
2.	2^{nd} derivative	f''	$\frac{d^2y}{dx^2}$	y''	$\frac{d^2}{dx^2}[f(x)]$
3.	3^{rd} derivative	f'''	$\frac{d^3y}{dx^3}$	y'''	$\frac{d^3}{dx^3}[f(x)]$
4.	4^{th} derivative	$f^{(4)}$	$\frac{d^4y}{dx^4}$	$y^{(4)}$	$\frac{d^4}{dx^4}[f(x)]$
5.	n^{th} derivative	$f^{(n)}$	$\frac{d^ny}{dx^n}$	$y^{(n)}$	$\frac{d^n}{dx^n}[f(x)]$

Caution: $\frac{d^n}{dx^n}$ **does <u>not</u> mean "d to the n^{th} power over d times x to the n^{th} power".**

III. Finding and Evaluating Higher-Order Derivatives

To calculate higher order derivatives, we will use the same differentiation rules that we used to calculate first derivatives.

Hint: If possible, simplify before differentiating.

Example 1: **Given $y = 2x^4 - 5x$, find $\frac{d^2y}{dx^2}$.**

$$\frac{dy}{dx} = 8x^3 - 5 \qquad \frac{d^2y}{dx^2} = 24x^2$$

Example 2: **Given $y = \sqrt[4]{x}$, find $\frac{d^2y}{dx^2}$.**

$$y = \sqrt[4]{x} = x^{\frac{1}{4}} \qquad \frac{dy}{dx} = \frac{1}{4}x^{-\frac{3}{4}} \qquad \frac{d^2y}{dx^2} = -\frac{3}{16}x^{-\frac{7}{4}} = -\frac{3}{16\sqrt[4]{x^7}}$$

Example 3: **Given $f(x) = (x^2 + 3x)^7$, find $f''(x)$.**

$$f'(x) = 7(x^2+3x)^6 \cdot (2x+3) = 7(2x+3)\cdot(x^2+3x)^6$$

$$f''(x) = 7(2x+3)\cdot 6(x^2+3x)^5 \cdot (2x+3) + (x^2+3x)^6 \cdot (14) = 7(2x+3)^2 \cdot 6(x^2+3x)^5 + 14(x^2+3x)^6$$

$$= 42\,(2x+3)^2 \cdot (x^2+3x)^5 + 14\,(x^2+3x)^6$$

Example 4: **Given $y = (x^3-2)\cdot(5x+1)$, find y''.**

Let's simplify y first: $y = (x^3-2)\cdot(5x+1) = 5x^4 + x^3 - 10x - 2$

$y' = 20x^3 + 3x^2 - 10$ $\qquad$ $y'' = 60x^2 + 6x$

Example 5: **For $y = x^7 - 8x^2 + 2$, find $\frac{d^6y}{dx^6}$.**

$\frac{dy}{dx} = 7x^6 - 16x$ $\qquad$ $\frac{d^2y}{dx^2} = 42x^5 - 16$ $\qquad$ $\frac{d^3y}{dx^3} = 210x^4$

$\frac{d^4y}{dx^4} = 840x^3$ $\qquad$ $\frac{d^5y}{dx^5} = 2520x^2$ $\qquad$ $\frac{d^6y}{dx^6} = 5040x$

IV. Velocity and Acceleration

The position function s(t) gives the position of an object at time t.

The derivative of the position function $s'(t)$ gives the instantaneous rate of change with respect to time. In other words, the derivative of the position function $s'(t)$ gives the velocity of the object at time t.

$$v(t) = s'(t) = \lim_{h\to 0} \frac{s(t+h)-s(t)}{h}.$$

The derivative of the velocity function (the second derivative of the position function) gives the instantaneous rate of change of the velocity. In other words, the derivative of velocity $v'(t)$ gives the acceleration of the object. $a(t) = v'(t) = s''(t)$.

<u>Units of Measure</u>

Position is measured in units of length such as feet, miles, meters, or kilometers.

Velocity is measured in units of length per unit of time such as ft per sec ft/sec and miles per hr mi/hr.

Acceleration is measured in units of length per unit of time per unit of time such as ft per sec per sec ft/sec^2 or meters per hour per hour m/hr^2.

Example 6: **Given the position function $s(t) = -10t^2 + 2t + 5$, where s is in meters and t is in seconds, find each of the following.**

a. v(t)

b. a(t)

c. The velocity and acceleration when t = 1 sec.

a. $v(t) = s'(t) = -20t + 2$

b. $a(t) = v'(t) = -20$

c. $v(1) = -20(1) + 2 = -18\ \text{m}/\text{sec}$ $\qquad$ $a(1) = -20\ \text{m}/\text{sec}^2$

V. Interpretations of Derivatives

A. First Derivative

The first derivative gives the instantaneous rate of change of the output variable with respect to the input variable at a specific value of the input variable. It also gives the slope of the line tangent to a function f at a specified point (x, f(x)). A real-life example of a first derivative is velocity which gives the instantaneous rate of change of position.

Examples:

"The train was travelling at 75 mph went it crossed the river $\frac{1}{2}$ an hour after leaving the station."

"The patient's temperature began decreasing 25 minutes after he took some Tylenol."

B. Second Derivative

The second derivative measures how the rate of change is itself changing. It tells whether the rate of change is speeding up or slowing down. A real-life example of a second derivative is acceleration which gives the instantaneous rate of change of velocity.

Examples:

"The rate at which the population of Bentonville was declining accelerated in May when many vendors closed their local offices."

"The U.S. economy grew at a slower rate in the second quarter."