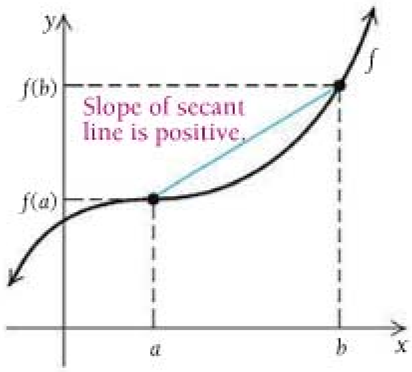
2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

I. Increasing and Decreasing Functions

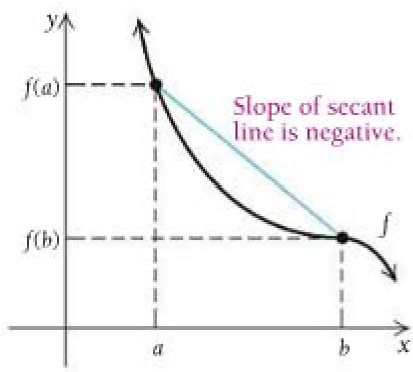
A. Increasing Functions

If the graph of a function rises from left to right over an interval I, the function is said to be **increasing** on, or over, I. This implies that for every *a* and *b* in I, if the input *a* is less than the input *b*, then the output for *a* is less than the output for *b*, i.e., if $a < b$, then $f(a) < f(b)$. In addition, for all *a* and *b* in I such that $a < b$, the slope of the secant line between x = a and x = b is positive.



B. Decreasing Functions

If the graph of a function drops from left to right over an interval I, the function is said to be **decreasing** on, or over, I. This implies that for every *a* and *b* in I, if the input *a* is less than the input *b*, then the output for *a* is greater than the output for *b*, i.e., if $a < b$, then $f(a) > f(b)$. In addition, for all *a* and *b* in I such that $a < b$, the slope of the secant line between x = a and x = b is negative.



C. Determining Increasing or Decreasing Behavior Using a Derivative: Theorem 1

If $f'(x) > 0$ for all x in an open interval I, then f is increasing over I.

If $f'(x) < 0$ for all x in an open interval I, then f is decreasing over I.

Note: An open interval means the end values are not included in the interval.

II. Critical Values

A **critical value** of a function f is any number c in the domain of f for which the tangent line at (c, f(c)) is horizontal or for which the derivative does not exist. That is, c is a critical value if f(c) exists and $f'(x) = 0$ or $f'(c)$ does not exist.

A continuous function can change from increasing to decreasing or vice versa only at a critical value.

III. Relative Maximum and Minimum Values

A. Relative Minimum

A **relative minimum** can be loosely thought of as the second coordinate of a "valley" on a graph. A relative minimum point is lower than all other points in the interval and its y-value is lower than that of points to the left and right of it.

More formally, if I is the domain of function f, f(c) is a relative minimum if there exists within I an open interval I_1 containing c such that $f(c) \le f(x)$, for all x in I_1.

The plural of minimum is minima.

B. Relative Maximum

A **relative maximum** can be loosely thought of as the second coordinate of a "peak" on a graph. A relative maximum point is higher than all other points in the interval and its y-value is higher than that of points to the left and right of it.

More formally, if I is the domain of function f, f(c) is a relative maximum if there exists within I an open interval I_2 containing c such that $f(c) \ge f(x)$, for all x in I_2.

The plural of maximum is maxima.

Collectively, relative minima and maxima are called **relative extrema** (singular: extremum).

Note: The second coordinates of the absolute highest and lowest points on a graph are called the **absolute maximum** and the **absolute minimum**.

C. Finding Relative Extrema: Theorem 2

If a function f has a relative extreme value f(c) on an open interval, then c is a critical value, so $f'(c) = 0$ or $f'(c)$ is undefined.

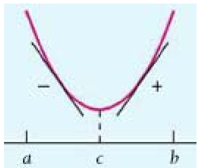
To find relative extrema, we need only consider those inputs for which the derivative is zero or for which the derivative is undefined.

We can think of a critical value as a candidate for a value where a relative extremum *might* occur. Not every critical value will yield a relative maximum or minimum. But if a relative maximum or minimum does occur, the first coordinate of the extremum will be a critical value.

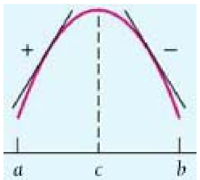
D. The First Derivative Test for Relative Extrema: Theorem 3

If f is continuous and differentiable on the interval (a, b) and c is the only critical value in the interval, then f(c) can be classified as a relative minimum, relative maximum, or neither as follows.

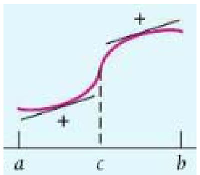
1. If f is decreasing to the left of x = c ($f'(x)$ is negative) and f is increasing to the right of x = c ($f'(x)$ is positive), then f(c) is a relative minimum.



2. If f is increasing to the left of x = c ($f'(x)$ is positive) and f is decreasing to the right of x = c, ($f'(x)$ is negative), then f(c) is a relative maximum.



3. If $f'(x)$ is the same sign to the left and right of x = c, then f(c) is not a relative extremum of f.



IV. Using the First Derivative to Graph a Function

1. Find all critical values by determining where $f'(x)$ is 0 and where $f'(x)$ is undefined (but f(x) is defined).
2. Use the critical values to divide the x-axis into intervals and choose a test value in each interval. Find the sign of $f'(x)$ for each test value and use this information to determine where f(x) is increasing or decreasing.
3. Classify any extrema as relative maxima or minima. Find the y-value of each extrema by plugging its x-value into the original function.
4. Plot some additional points and sketch the graph.

Note: The derivative is used to find the critical values of f and test values are plugged into $f'(x)$. Y-values of points are found by plugging an x into the original function.

Example 1: **Find the relative extrema of the function $G(x) = x^3 - x^2 - x + 2$, if they exist. List each extremum along with the x-value at which it occurs. Then sketch a graph of the function.**

1. $G'(x) = 3x^2 - 2x - 1$ $G'(x)$ is a polynomial and as such, is never undefined.

$3x^2 - 2x - 1 = 0 \rightarrow (3x + 1)(x - 1) = 0 \rightarrow x = -\frac{1}{3}$ and $x = 1$ are critical values.

2.

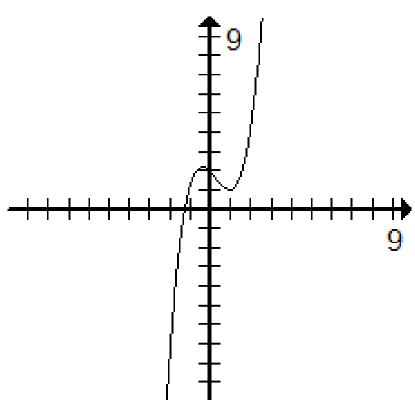
Interval	**Test Value**	**Sign of $G'(x)$**	**Result**
$\left(-\infty, -\frac{1}{3}\right)$	–1	$G'(-1) = 4 > 0$	increasing
$\left(-\frac{1}{3}, 1\right)$	0	$G'(0) = -1 < 0$	decreasing
$(1, \infty)$	2	$G'(2) = 7 > 0$	increasing

3. $G\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 - \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) + 2 = \frac{59}{27}$ $\rightarrow$ There is a relative max at $\left(-\frac{1}{3}, \frac{59}{27}\right)$.

$G(1) = (1)^3 - (1)^2 - 1 + 2 = 1$ $\rightarrow$ There is a relative min at (1, 1).

4.

x	G(x)
–2	–8
–1	1
0	2
2	4
3	17



Example 2: **Find the relative extrema of the function $G(x)=\sqrt[3]{x+2}$, if they exist. List each extremum along with the x-value at which it occurs. Then sketch a graph of the function.**

1. $G(x)=\sqrt[3]{x+2}=(x+2)^{\frac{1}{3}}$

$$G'(x)=\tfrac{1}{3}(x+2)^{-\frac{2}{3}}\cdot 1=\tfrac{1}{3}(x+2)^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{(x+2)^2}}$$

$G'(x)=0$ has no solution. Since there is a constant in the numerator, there is no way for the fraction to equal 0.

$G'(x)$ is undefined when the denominator of the fraction equals zero, i.e. when x = –2.

The only critical value is x = –2.

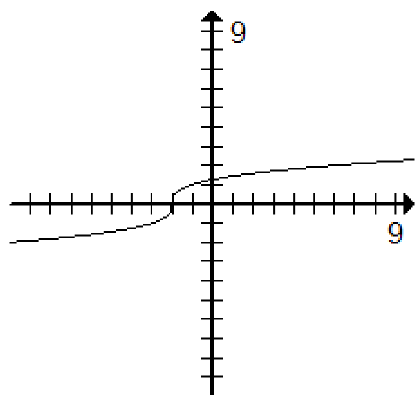
2.

Interval	**Test Value**	**Sign of $G'(x)$**	**Result**
$(-\infty,-2)$	–3	$G'(-3)=\frac{1}{3}>0$	increasing
$(-2,\infty)$	0	$G'(0)=\frac{1}{3\sqrt[3]{4}}\approx .210>0$	increasing

3. There is no relative minimum or maximum.

4.

x	G(x)
–10	–2
–3	–1
–2	0
–1	1
6	2



Example 3: **Find the relative extrema of the function $G(x)=-\dfrac{8}{x^2+1}$, if they exist. List each extremum along with the x-value at which it occurs. Then sketch a graph of the function.**

1. $G(x)=-\dfrac{8}{x^2+1}=-8(x^2+1)^{-1}$

$$G'(x)=-8\left(-1(x^2+1)^{-2}\cdot 2x\right)=16x\left(x^2+1\right)^{-2}=\frac{16x}{\left(x^2+1\right)^2}$$

Since there is no real solution to the equation $x^2 + 1 = 0$, there are no x-values at which $G'(x)$ is undefined.

$G'(x)$ = 0 when x = 0, so the only critical value is x = 0.

2.

Interval	**Test Value**	**Sign of** $G'(x)$	**Result**
$(-\infty, 0)$	–1	$G'(-1) = -4 < 0$	decreasing
$(0, \infty)$	1	$G'(1) = 4 > 0$	increasing

3. $G(0) = -\frac{8}{(0)^2 + 1} = -8$ There is a relative minimum at (0, –8).

4.

x	G(x)
–3	$-\frac{4}{5}$
–2	$-\frac{8}{5}$
–1	–4
1	–4
2	$-\frac{8}{5}$
3	$-\frac{4}{5}$