

## 2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

### I. Concavity: Increasing and Decreasing Derivatives

#### A. Definition

A curve that curls upward (“holds water”) is said to be **concave up**.

A curve that curls downward (“spills water”) is said to be **concave down**.

More formally, if  $f$  is a function whose derivative  $f'$  exists at every point in an open interval  $I$ , then  $f$  is concave up on  $I$  if  $f'$  is increasing over  $I$  and  $f$  is concave down on  $I$  if  $f'$  is decreasing over  $I$ .

#### B. A Test for Concavity: Theorem 4

1. If  $f''(x) > 0$  on an interval  $I$ , then the graph of  $f$  is concave up.
2. If  $f''(x) < 0$  on an interval  $I$ , then the graph of  $f$  is concave down.

#### C. Distinguishing Between Slope and Concavity

Slope is a measure of steepness. Concavity is a measure of curl. All combinations of slope and concavity are possible.

$f'(x) < 0$  and  $f''(x) < 0$ : decreasing, concave down

$f'(x) < 0$  and  $f''(x) > 0$ : decreasing, concave up

$f'(x) > 0$  and  $f''(x) < 0$ : increasing, concave down

$f'(x) > 0$  and  $f''(x) > 0$ : increasing, concave up

### II. Classifying Relative Extrema Using Second Derivatives

#### A. The Second-Derivative Test for Relative Extrema: Theorem 5

Suppose that  $f$  is differentiable for every  $x$  in an open interval  $(a, b)$  and  $c$  is a critical value in the interval.

1. If  $f'(x) = 0$  and  $f''(x) > 0$ , then the graph is concave up around  $c$  and  $f(c)$  is a relative minimum.
2. If  $f'(x) = 0$  and  $f''(x) < 0$ , then the graph is concave down around  $c$  and  $f(c)$  is a relative maximum.

#### B. Finding and Classifying Relative Extrema

1. Find  $f'(x)$  and  $f''(x)$ .
2. Find the critical values where  $f'(x) = 0$  or  $f'(x)$  is undefined.
3. Identify the critical points by finding the  $y$ -value associated with each critical value.  
**Hint:** Plug into the original function to find  $y$ -values of points.
4. Apply the Second Derivative Test by plugging each critical value into  $f''(x)$  and determining if the result is positive (the critical point is a relative minimum) or negative (the critical point is a relative maximum).

**Example 1:** For the function  $f(x) = 8x^3 - 6x + 1$ , find all relative extrema and classify each as a maximum or minimum. Use the Second Derivative Test where possible.

1.  $f'(x) = 24x^2 - 6$                        $f''(x) = 48x$
2. Since  $f'(x)$  exists for all real numbers, the only critical values occur when  $f'(x) = 0$ .  
 $24x^2 - 6 = 0 \rightarrow 24x^2 = 6 \rightarrow x^2 = \frac{1}{4} \rightarrow x = \pm \frac{1}{2}$  (critical values)
3.  $f\left(-\frac{1}{2}\right) = 8\left(-\frac{1}{2}\right)^3 - 6\left(-\frac{1}{2}\right) + 1 = 3$     One critical point is  $\left(-\frac{1}{2}, 3\right)$ .  
 $f\left(\frac{1}{2}\right) = 8\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right) + 1 = -1$     The second critical point is  $\left(\frac{1}{2}, -1\right)$
4.  $f''\left(-\frac{1}{2}\right) = 48\left(-\frac{1}{2}\right) = -24$      $-24 < 0$     Therefore 3 is a relative maximum when  $x = -\frac{1}{2}$ .  
 $f''\left(\frac{1}{2}\right) = 48\left(\frac{1}{2}\right) = 24$      $24 > 0$     Therefore  $-1$  is a relative minimum when  $x = \frac{1}{2}$ .

**Note:** If  $c$  is a critical value and  $f''(x) = 0$ , the First Derivative Test must be used to classify  $f(c)$ .

### III. Points of Inflection

#### 1. Definition

A point where the concavity of a graph changes from concave up to concave down, or vice versa, is called an **inflection point**.

#### 2. Finding Points of Inflection: Theorem 6

If a function  $f$  has a point of inflection, it must occur at a point  $x_0$ , where  $f''(x_0) = 0$  or  $f''(x_0)$  is undefined.

#### 3. How to Find the Inflection Points of a Curve

- a. Find  $f''(x)$ .
- b. Find the  $x$ -values where  $f''(x) = 0$  or where  $f''(x)$  is undefined. Exclude any  $x$ -values which are not in the domain of  $f$ .
- c. Find the  $y$ -values of the possible inflection points (PIPs) by plugging each  $x$  from step b into the original function.
- d. Use the  $x$ -values of the PIPs to break the number line into intervals. Test a value from each interval by plugging it into  $f''(x)$  and determining its sign. Identify each interval as concave up or concave down.
- e. Label each point where the concavity changes from up to down or vice versa as an inflection point.

**Example 2:** Use the second derivative to find the points of inflection for the function in Example 2.

- a.  $f(x) = 8x^3 - 6x + 1$                        $f'(x) = 24x^2 - 6$                        $f''(x) = 48x$
- b.  $f''(x)$  is defined for all real numbers, so the only  $x$ -values of PIPs must come from setting  $f''(x) = 0$ .  
 $48x = 0 \rightarrow x = 0$
- c.  $f(0) = 8(0)^3 - 6(0) + 1 = 1$     So the only PIP is  $(0, 1)$ .

d.

Interval	Test Value	Sign of $f''(x)$	Result
$(-\infty, 0)$	-1	$f''(-1) = -48 < 0$	concave down
$(0, \infty)$	1	$f''(1) = 48 > 0$	concave up

e. Since the concavity changed from concave down to concave up at  $x = 0$ ,  $(0, 1)$  is an inflection point.

#### IV. Strategy for Sketching Graphs

##### 1. Derivatives and Domain

Find  $f'(x)$  and  $f''(x)$ . Note the domain of  $f$ .

##### 2. Critical Values of $f$

Find the critical values by solving  $f'(x) = 0$  and finding where  $f'(x)$  does not exist. These numbers yield candidates for relative maxima or minima. Find the function values at these points.

##### 3. Increasing and/or Decreasing Intervals and Relative Extrema

Substitute each critical value,  $c$ , from step 2 into  $f''(x)$ . If  $f''(c) < 0$ , then  $f(c)$  is a relative maximum and  $f$  is increasing to the left of  $c$  and decreasing to the right. If  $f''(c) > 0$ , then  $f(c)$  is a relative minimum and  $f$  is decreasing to the left of  $c$  and increasing to the right.

**Note: If the Second Derivative Test fails, we must use the First Derivative Test to identify relative extrema.**

##### 4. Inflection Points

Determine candidates for inflection points by finding where  $f''(x) = 0$  or where  $f''(x)$  does not exist. Find the function values at these points. These are our possible inflection points (PIPs).

##### 5. Concavity

Use the  $x$ -values of the PIPs to break the number line into intervals. Substitute test values into  $f''(x)$  to determine where the graph is concave up ( $f''(x) > 0$ ) and where it is concave down ( $f''(x) < 0$ ).

6. Find some additional points. Then use those, along with the information from steps 1 – 5 (relative extrema, location of increasing & decreasing intervals, inflection points, location of intervals which are concave up & concave down), to create an accurate sketch of the graph.

**Example 3: Sketch the graph of the function  $f(x) = 80 - 9x^2 - x^3$ . List the coordinates of where extrema or points of inflection occur. State where the function is increasing or decreasing, as well as where it is concave up or concave down.**

1.  $f'(x) = -18x - 3x^2$        $f''(x) = -18 - 6x$       Domain of  $f$ :  $(-\infty, \infty)$

2. Since  $f'(x)$  is a polynomial with a domain of all real numbers, it is never undefined.

$$-18x - 3x^2 = 0 \rightarrow -3x(6 + x) = 0 \rightarrow x = 0 \text{ and } x = -6 \text{ (critical values)}$$

$$f(-6) = 80 - 9(-6)^2 - 3(-6)^3 = -28 \quad \text{The first critical point is } (-6, -28).$$

$$f(0) = 80 - 9(0)^2 - 3(0)^3 = 80 \quad \text{The second critical point is } (0, 80).$$

3.  $f''(-6) = -18 - 6(-6) = 18$      $18 > 0$      $-28$  is a relative minimum when  $x = -6$

$$f''(0) = -18 - 6(0) = -18$$

decreasing:  $(-\infty, -6)$     increasing:  $(-6, 0)$     decreasing:  $(0, \infty)$

4. Since  $f''(x)$  is a polynomial with a domain of all real numbers, it is never undefined.

$$-18 - 6x = 0 \rightarrow -18 = 6x \rightarrow x = -3$$

$$f(-3) = 80 - 9(-3)^2 - (-3)^3 = 26 \quad (-3, 26) \text{ is a possible inflection point}$$

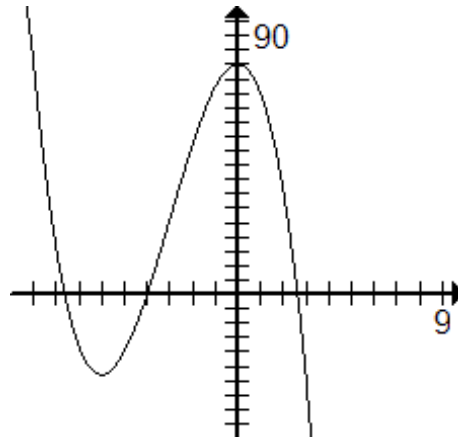
- 5.

Interval	Test Value	Sign of $f''(x)$	Result
$(-\infty, -3)$	-4	$f''(-4) = 6 > 0$	concave up
$(-3, \infty)$	0	$f''(0) = -18 < 0$	concave down

Since the concavity changed from concave up to concave down at  $x = -3$ ,  $(-3, 26)$  is an inflection point.

- 6.

x	y	
-8	16	
-6	-28	rel min
-4	0	
-3	26	I.P.
-2	52	
0	80	rel max
2	36	



**Example 4:** Sketch the graph of the function  $f(x) = (x-1)^{\frac{2}{3}}$ . List the coordinates of where extrema or points of inflection occur. State where the function is increasing or decreasing, as well as where it is concave up or concave down.

1.  $f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x-1}}$        $f''(x) = -\frac{2}{9}(x-1)^{-\frac{4}{3}} = \frac{-2}{9\sqrt[3]{(x-1)^4}}$       Domain of  $f$ :  $(-\infty, \infty)$

2. Since  $f'(x)$  is a fraction with a constant in the numerator, it can never equal zero.  $f'(x)$  will be undefined when the denominator equals zero which will only happen if  $x = 1$ .

$$f(1) = (1-1)^{\frac{2}{3}} = 0 \quad \text{Our only critical point is } (1, 0).$$

3. If we plug 1 into  $f''(x)$ , it is undefined, so we cannot use the Second Derivative Test. Therefore we must use the First Derivative Test. We use 1 to divide the number line into two intervals and then we test a number in each interval.

Interval	Test Value	Sign of $f'(x)$	Result
$(-\infty, 1)$	0	$f'(0) = -\frac{2}{3} < 0$	decreasing
$(1, \infty)$	2	$f'(2) = \frac{2}{3} > 0$	increasing

Since the slope changed from decreasing to increasing 0 is a relative min when  $x = 1$ .

4. Since  $f''(x)$  is a fraction with a constant in the numerator, it can never equal zero.  $f''(x)$  will be undefined when the denominator equals zero which will only happen if  $x = 1$ .  
 $f(1) = (1-1)^{\frac{2}{3}} = 0$  Our only possible inflection point is  $(1, 0)$ .

5.

Interval	Test Value	Sign of $f''(x)$	Result
$(-\infty, 1)$	0	$f''(0) = -\frac{2}{9} < 0$	concave down
$(1, \infty)$	2	$f''(2) = -\frac{2}{9} < 0$	concave down

Since the concavity did not change at  $x = 1$ , the point  $(1, 0)$  is not an inflection point.

6.

x	y	
-2	2.0801	
-1	1.5874	
0	1	
1	0	rel min
2	1	
3	1.5874	
4	2.0801	

