2.4 Using Derivatives to Find Absolute Maximum and Minimum Values

I. Absolute Maximum and Minimum Values

Suppose that f is a function with domain I.

f(c) is an **absolute minimum** if $f(c) \le f(x)$ for all x in I.
In other words, an absolute minimum is the smallest value of the function over its entire domain.

f(c) is an **absolute maximum** if $f(c) \ge f(x)$ for all x in I.
In other words, an absolute maximum is the largest value of the function over its entire domain.

II. The Extreme-Value Theorem

A continuous function f defined over a closed interval [a, b] must have an absolute maximum value and an absolute minimum value over [a, b].

Note: A closed interval such as [a, b] includes the end values, whereas an open interval such as (a, b) does not.

III. Finding Absolute Maximum and Minimum Values over Closed Intervals

Maximum–Minimum Principle 1 (Theorem 8)

Suppose that f is a continuous function over a closed interval [a, b]. To find the absolute maximum and minimum values over [a, b]:

1. First find $f'(x)$.
2. Then find all critical values in [a, b]. That is, find all c values in [a, b] for which $f'(c) = 0$ or for which $f'(c)$ does not exist.
3. List the critical values from step 2 and the endpoints of the interval: $a, c_1, c_2, ..., c_n, b$.
4. Evaluate f(x) for each value in step 3: $f(a), f(c_1), f(c_2), ..., f(c_n), f(b)$.

   The largest of these is the absolute maximum of f over [a, b].
   The smallest of these is the absolute minimum of f over [a, b].

Note: Endpoints of a closed interval can be absolute extrema but *not* relative extrema.

Example 1: Find the absolute maximum and minimum values of the function over the indicated interval, and indicate the x-values at which they occur.

$f(x) = x^3 + \frac{1}{2}x^2 - 2x + 4$; [–2, 0]

1. $f'(x) = 3x^2 + x - 2$
2. $f'(x)$ is never undefined

   $3x^2 + x - 2 = 0 \quad \rightarrow \quad (3x - 2)(x + 1) = 0 \quad \rightarrow \quad x = \frac{2}{3};\ x = -1$
3. $\frac{2}{3}$ is not in the interval [–2, 0], so the only x-values where extrema might occur are: –2, –1, 0
4. $f(-2) = (-2)^3 + \frac{1}{2}(-2)^2 - 2(-2) + 4 = 2$ Absolute minimum: 2 at x = –2

   $f(-1) = (-1)^3 + \frac{1}{2}(-1)^2 - 2(-1) + 4 = 5.5$ Absolute maximum: 5.5 at x = –1

   $f(0) = (0)^3 + \frac{1}{2}(0)^2 - 2(0) + 4 = 4$

IV. Finding Absolute Extrema When There Is Only One Critical Value in the Interval

Maximum–Minimum Principle 2 (Theorem 9)

Suppose that f is a function such that $f'(x)$ exists for every x in an interval I and that there is *exactly one* critical value c in I for which $f'(c) = 0$. Then

f(c) is the absolute maximum value over I if $f''(c) < 0$ **OR**

f(c) is the absolute minimum value over I if $f''(c) > 0$.

Note: Theorem 9 holds no matter what the interval I is – whether open, closed, or infinite in length.

If $f''(c) = 0$, either we must use Maximum–Minimum Principle 1, or we must know more about the the behavior of the function over the given interval.

V. A General Strategy for Finding Absolute Extrema

1. Find $f'(x)$.
2. Find the critical values by finding the c values where $f'(c) = 0$ or for which $f'(c)$ does not exist.
3. If the interval is closed and there is more than one critical value, use Maximum–Minimum Principle 1. That is, find the absolute extrema by plugging all critical values in the interval and the end values of the interval into the original function.
4. If the interval is closed and there is exactly one critical value, use either Maximum–Minimum Principle 1 or Principle 2. If it is easy to find $f''(x)$, use Principle 2. That is, plug the one critical value into $f''(x)$ and observe whether the result is positive or negative. If $f''(c) < 0$, f(c) is the absolute maximum and if $f''(c) > 0$, f(c) is the absolute minimum.
5. If the interval is not closed, such as $(-\infty, \infty)$, $(0, \infty)$, or (a, b), and the function has only one critical value, use Maximum–Minimum Principle 2. In such a case, if the function has a maximum, it will have no minimum; and if it has a minimum, it will have no maximum.

Example 2: Find the absolute maximum and minimum values of the function, if they exist, over the indicated interval. Also indicate the x-value at which each extremum occurs. If no interval is specified, use all real numbers $(-\infty, \infty)$.

$f(x) = 16x - \frac{4}{3}x^3$; $(0, \infty)$

1. $f'(x) = 16 - 4x^2$
2. $f'(x)$ is never undefined

   $16 - 4x^2 = 0 \rightarrow 16 = 4x^2 \rightarrow 4 = x^2 \rightarrow x = -2$ or $x = 2$
3. Since –2 is not in the interval $(0, \infty)$, there is only one critical value in the interval. Therefore we can use Maximum–Minimum Principle 2.

   $f''(x) = -8x \rightarrow f''(2) = -16 < 0$
4. Therefore f(2) = $21\frac{1}{3}$ is the absolute maximum and it occurs when x = 2.

Example 3: **For a dosage of x cubic centimeters (cc) of a certain drug, the resulting blood pressure B is approximated by**

$B(x) = 305x^2 - 1830x^3, \quad 0 \le x \le .16.$

Find the maximum blood pressure and the dosage at which it occurs.

1. $B'(x) = 610x - 5490x^2$

2. $B'(x)$ is never undefined.

   $610x - 5490x^2 = 0 \quad \rightarrow \quad 610x\,(1 - 9x) = 0 \quad \rightarrow \quad x = 0,\ x = \frac{1}{9}$

   Since we have two critical values, we will use Maximum–Minimum Principle 1.

3. x-values where the maximum may occur: 0, $\frac{1}{9}$, .16

4. $B(0) = 305(0)^2 - 1830(0)^3 = 0$

   $B\left(\frac{1}{9}\right) = 305\left(\frac{1}{9}\right)^2 - 1830\left(\frac{1}{9}\right)^3 = 1.25514$

   $B(.16) = 305(.16)^2 - 1830(.16)^3 = .31232$

   Maximum blood pressure is 1.25514 when x = $\frac{1}{9}$ cc.