2.5 Maximum–Minimum Problems; Business and Economics Applications

I. A General Strategy for Solving Optimization Applications

1. Read the problem very carefully. If possible, draw a picture.
2. Identify the unknowns and denote each with a meaningful variable. Label your picture with the appropriate variables and constants, noting what varies, what stays fixed, and what units are used.
3. Identify the variable that is to be maximized or minimized and express it as a function of the other variables. We will call this the primary equation. If this equation has only one independent variable, go straight to step 6.
4. If the primary equation has more than one independent variable, reread the problem and search for a relationship between the variables. Write an equation that expresses this relationship. We will call this the secondary equation.
5. Solve the secondary equation for one independent variable and substitute this expression into the primary equation. The primary equation should now have only one independent variable.
6. Determine the feasible domain of the primary equation – the x values which make sense in the context of the problem.
7. Take the derivative of the primary function and find its critical values. Exclude any CV which are not in the feasible domain. If the feasible domain is a closed interval, its endpoints should be considered as possible critical values.
8. Find the second derivative of the primary equation. Plug the CV into the second derivative to determine whether there is a maximum or a minimum.
9. Find the maximum or minimum by plugging the CV into the primary equation. Solve for other variables, as needed.

II. Useful Formulas and Relationships

1. Rectangles

   $P = 2L + 2W$ $A = L \cdot W$

2. Rectangular Solids (Boxes)

   $V = L \cdot W \cdot H$ $SA = 2LW + 2LH + 2WH$ [closed box]

3. Cylindrical Solids (Cans)

   $V = \pi \cdot r^2 \cdot h$ $SA = 2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot h$

4. Cost, Revenue, and Profit

   Total Revenue = price per unit · number of units → $R(x) = p \cdot x$

   Total Profit = Total revenue – Total cost → $P(x) = R(x) - C(x)$

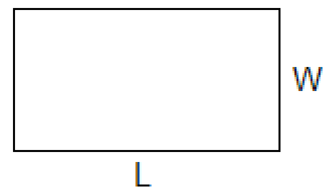
5. In optimization problems where a change will make one value rise and another fall, it is often easiest to choose x to be the number of such changes. In such cases, a negative x indicates a decrease.
6. Theorem 10

   Maximum profit occurs at those x-values for which $R'(x) = C'(x)$ and $R''(x) < C''(x)$.

Example 1: Maximizing Area

Of all rectangles that have a perimeter of 42 ft, find the dimensions of the one with the largest area. What is the area?

W = width L = length



primary equation: $A = L \cdot W$

secondary equation: $2L + 2W = 42$

$2L = 42 - 2W$

$L = 21 - W$

$A = (21 - W) \cdot W = 21W - W^2$ Feasible domain: (0, 21)

$A = 21W - W^2 \rightarrow A' = 21 - 2W \rightarrow 21 - 2W = 0 \rightarrow 21 = 2W \rightarrow W = 10.5$ ft

$A'' = -2 < 0$ Therefore, the maximum area occurs when W = 10.5 ft.

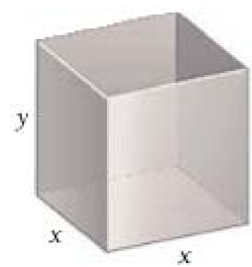
$L = 21 - 10.5 = 10.5$ ft

Maximum area: $10.5 \times 10.5 = 110.25 \text{ ft}^2$

Example 2: Minimizing Surface Area

A soup company is constructing an open-top, square-based, rectangular metal tank that will have a volume of 32 ft^3. What dimensions will minimize the surface area? What is the minimum surface area?

x = length x = width y = height



primary equation: $SA = x^2 + 4xy$

secondary equation: $x^2 \cdot y = 32$

$y = \frac{32}{x^2}$

$SA = x^2 + 4x \cdot \frac{32}{x^2} = x^2 + \frac{128}{x} = x^2 + 128x^{-1}$ Feasible domain: (0, 32)

$SA = x^2 + 128x^{-1} \rightarrow SA' = 2x - 128x^{-2} \rightarrow 2x - 128x^{-2} = 0 \rightarrow$

$2x \cdot (x^2) - 128x^{-2} \cdot (x^2) = 0 \cdot (x^2) \rightarrow 2x^3 - 128 = 0 \rightarrow 2x^3 = 128 \rightarrow x^3 = 64$

$x = \sqrt[3]{64} = 4$ ft $SA'' = 2 + 256x^{-3}$ $SA''(4) = 2 + 256(4)^{-3} = 2 + 4 = 6 > 0$

Therefore, the minimum surface area occurs when x = 4 ft.

$y = \frac{32}{4^2} = \frac{32}{16} = 2$ ft Minimum surface area: $SA = (4)^2 + 4 \cdot 4 \cdot 2 = 16 + 32 = 48 \text{ ft}^2$

Example 3: Maximizing Profit

Riverside Appliances is marketing a new refrigerator. It determines that in order to sell x refrigerators, the price per refrigerator must be p = 280 – .4x. It also determines that the total cost of producing x refrigerators is given by $C(x) = 5000 + .6x^2$.

a. Find the total revenue, R(x).

$R(x) = p \cdot x = (280 - .4x) \cdot x \quad \rightarrow \quad R(x) = 280x - .4x^2$

b. Find the total profit, P(x).

$P(x) = R(x) - C(x) = 280x - .4x^2 - (5000 + .6x^2) = 280x - .4x^2 - 5000 - .6x^2$

$P(x) = -x^2 + 280x - 5000$

Feasible domain: (0, 280) [Set P(x) = 0 & solve for x. $x \approx 261$ or $x \approx 19$]

c. How many refrigerators must the company produce and sell in order to maximize profit?

$P'(x) = -2x + 280 \quad \rightarrow \quad -2x + 280 = 0 \quad \rightarrow \quad -2x = -280 \quad \rightarrow \quad x = 140$

$P''(x) = -2 < 0$ Therefore, the maximum profit occurs when x = 140 refrigerators.

d. What is the maximum profit?

$P(140) = -(140)^2 + 280(140) - 5000 = \$14,600$

e. What price per refrigerator must be charged in order to maximize profit?

p = 280 – .4(140) = $224 per refrigerator

Example 4: Maximizing Yield

An orange grower finds that if he plants 80 orange trees per acre, each tree will yield 60 bushels of oranges. He estimates that for each additional tree that he plants per acre, the yield of each tree will decrease by 2 bushels. How many trees should he plant per acre to maximize his harvest?

x = the number of added trees per acre

With x extra trees per acre,

trees per acre: 80 + x [original 80 plus x more]

yield per tree: 60 – 2x [original yield minus 2 bushels per extra tree]

total yield per acre: Y(x) = (60 – 2x) (80 + x) [yield per tree times trees per acre]

$Y(x) = 4800 - 100x - 2x^2$

Feasible domain: (–80, 30) [Set Y(x) = 0 & solve for x. x = –80 or x = 30]

$Y'(x) = -100 - 4x \quad \rightarrow \quad -100 - 4x = 0 \quad \rightarrow \quad -100 = 4x \quad \rightarrow \quad x = -25$

$Y''(x) = -4 < 0$ Therefore, the maximum yield occurs when x = –25

The negative x value indicates that 25 *fewer* trees should be planted per acre.

The orange grower should plant 80 – 25 = 55 trees per acre to maximize his total yield per acre to $4800 - 100(-25) - 2(-25)^2 = 6050$ bushels per acre.

The text contains examples of maximizing volume, minimizing surface area of a can, and maximizing revenue.