2.7 Implicit Differentiation and Related Rates

I. Explicit versus Implicit Definition of a Function

A. Explicit Definition

A function written in the form y = f(x) is said to be **defined explicitly** because y is defined by a rule or formula in x alone. In other words, y is isolated on one side of the equation.

Examples: $y = 3x + 7$ $f(x) = 2x^3 – 4x^2 + 7x + 9$

B. Implicit Definition

A function is **defined implicitly** if y is defined by an equation in x and y. In such a case, it may be cumbersome or nearly impossible to isolate the y.

Examples: $x^2 + y^2 = 9$ $x^2y + 3xy + 2xy^2 = x^3$

II. Implicit Differentiation

A. Introduction

When it is difficult or impossible to isolate y in a function, we must use **implicit differentiation** to find the derivative of that function. Assuming x is the independent variable and y is the dependent variable, implicit differentiation will involve differentiating both sides of the equation with respect to x. This will involve using the Extended Power Rule:

$$\frac{d}{dx}y^n = n \cdot y^{n-1} \cdot \frac{dy}{dx}$$

B. Finding $\frac{dy}{dx}$ by Implicit Differentiation

1. Differentiate both sides of the equation with respect to x. **Make sure that the derivative of any term involving y includes the factor $\frac{dy}{dx}$.**
2. Collect all terms with $\frac{dy}{dx}$ on the left and all other terms on the right. The expression on the right may contain x and y.
3. Factor out and isolate $\frac{dy}{dx}$.

C. Finding the Slope of a Tangent Line

To determine the slope of a tangent line at a point on the graph of an implicit relationship, we may need to evaluate the derivative by inserting both the x-value and the y-value at the point of tangency.

Example 1: Differentiate $3x^3 – y^2 = 8$ implicitly to find $\frac{dy}{dx}$. Then find the slope of the curve at the point (2, 4).

$$\frac{d}{dx}\left(3x^3 - y^2\right) = \frac{d}{dx}(8) \quad \rightarrow \quad 9x^2 - 2y\frac{dy}{dx} = 0 \quad \rightarrow \quad -2y\frac{dy}{dx} = -9x^2 \quad \rightarrow \quad \frac{dy}{dx} = \frac{9x^2}{2y}$$

Slope of the curve at (2, 4): $\frac{dy}{dx} = \frac{9(2)^2}{2(4)} = \frac{9 \cdot 4}{8} = \frac{9}{2} = 4.5$

Example 2: Differentiate $x^4 - x^2 \cdot y^3 = 12$ implicitly to find $\frac{dy}{dx}$. Then find the slope of the curve at the point (–2, 1).

$$\frac{d}{dx}\left(x^4 - x^2 \cdot y^3\right) = \frac{d}{dx}(12) \quad \rightarrow \quad 4x^3 - \left(2x \cdot y^3 + 3y^2 \frac{dy}{dx} \cdot x^2\right) = 0 \quad \rightarrow$$

$$4x^3 - 2x \cdot y^3 - 3y^2 \frac{dy}{dx} \cdot x^2 = 0 \quad \rightarrow \quad -3x^2y^2 \frac{dy}{dx} = -4x^3 + 2xy^3 \quad \rightarrow$$

$$\frac{dy}{dx} = \frac{-4x^3+2xy^3}{-3x^2y^2} = \frac{4x^3-2xy^3}{3x^2y^2} = \frac{4x^2-2y^3}{3xy^2}$$

Slope of the curve at (–2, 1): $\frac{dy}{dx} = \frac{4(-2)^2-2(1)^3}{(3)(-2)(1)^2} = \frac{16-2}{-6} = \frac{14}{-6} = -\frac{7}{3}$

Example 3: Differentiate $x^3y^2 + x^5y^3 = -19$ implicitly to find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(x^3 \cdot y^2 + x^5 \cdot y^3\right) = \frac{d}{dx}(-19) \quad \rightarrow \quad \left(3x^2 \cdot y^2 + 2y \frac{dy}{dx} \cdot x^3\right) + \left(5x^4 \cdot y^3 + 3y^2 \frac{dy}{dx} \cdot x^5\right) = 0 \quad \rightarrow$$

$$2x^3y \frac{dy}{dx} + 3x^5y^2 \frac{dy}{dx} = -3x^2y^2 - 5x^4y^3 \quad \rightarrow \quad \frac{dy}{dx}\left(2x^3y + 3x^5y^2\right) = -3x^2y^2 - 5x^4y^3 \quad \rightarrow$$

$$\frac{dy}{dx} = \frac{-3x^2y^2-5x^4y^3}{2x^3y+3x^5y^2} = \frac{-3y-5x^2y^2}{2x+3x^3y}$$

III. Demand Equation

In economics, a demand equation is the relationship between the price p of an item and the quantity x that consumers will demand at that price. In general, price and quantity are inversely related: increasing the price decreases demand/sales and decreasing the price increases demand/sales.

Example 4: For the demand equation $x^3 \cdot p^2 = 108$, differentiate implicitly to find $\frac{dp}{dx}$.

$$\frac{d}{dx}\left(x^3 \cdot p^2\right) = \frac{d}{dx}(108) \quad \rightarrow \quad 3x^2 \cdot p^2 + 2p\frac{dp}{dx} \cdot x^3 = 0 \quad \rightarrow \quad 2x^3p\frac{dp}{dx} = -3x^2p^2 \quad \rightarrow$$

$$\frac{dp}{dx} = \frac{-3x^2p^2}{2x^3p} = \frac{-3p}{2x}$$

IV. Related Rates

A. Introduction

Sometimes both variables in an equation are functions of a third variable, usually t for time. In such problems, the rate at which x is changing with respect to time is related to the rate of change of y with respect to time. This is why we call these problems **related rates** problems.

Example:

Suppose that x and y are related by the equation $y = x^2 + 3$. If both variables are changing with respect to time, then their rates of change are related by the equation $\frac{dy}{dt} = 2x\frac{dx}{dt}$ which we obtained by implicitly differentiating $y = x^2 + 3$.

B. Guidelines for Solving Related Rate Problems

1. Assign a variable to each quantity. Draw a picture if possible.
2. Write the given values of the variable and their rates of change with respect to time.

Examples:

a. After traveling 1 hour, the velocity of a car is 50 mph.
x = distance traveled Therefore, $\frac{dx}{dt}$ = 50 mph when t = 1 hour.

b. After 6 months, revenue is increasing at the rate of $4000 per month.
R = revenue Therefore, $\frac{dR}{dt}$ = 4000 dollars per month when t = 6 months.

3. Write an equation expressing the relationship between the variables.
4. Differentiate both sides of the equation implicitly with respect to time (t).
5. Replace the variables and their derivatives with their given or calculated values and solve the equation for the required rate of change.

Caution: Differentiate first, then substitute values in for the variables.

Example 5: Find the rates of change of total revenue, cost, and profit with respect to time. Assume that R(x) and C(x) are in dollars.

$R(x) = 50x - .5x^2$ C(x) = 10x + 3 when x = 10 and $\frac{dx}{dt} = 5$ units per day

$R(x) = 50x - .5x^2 \rightarrow \frac{d}{dt}[R(x)] = \frac{d}{dt}(50x - .5x^2) \rightarrow \frac{dR}{dt} = 50\frac{dx}{dt} - x\frac{dx}{dt} \rightarrow$

$\frac{dR}{dt} = 50(5) - (10)(5) = 250 - 50 = \$200$ per day

$C(x) = 10x + 3 \rightarrow \frac{d}{dt}[C(x)] = \frac{d}{dt}(10x + 3) \rightarrow \frac{dC}{dt} = 10\frac{dx}{dt} \rightarrow$

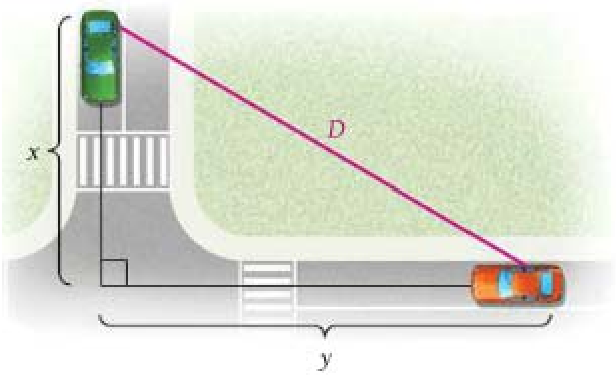
$\frac{dC}{dt} = (10)(5) = \$50$ per day

$P(x) = R(x) - C(x) = 50x - .5x^2 - (10x + 3) = -.5x^2 + 40x - 3$

$\frac{d}{dt}[P(x)] = \frac{d}{dt}(-.5x^2 + 40x - 3) \rightarrow \frac{dP}{dt} = -x\frac{dx}{dt} + 40\frac{dx}{dt} \rightarrow$

$\frac{dP}{dt} = -10(5) + 40(5) = -50 + 200 = \$150$ per day

Example 6: Two cars start from the same point at the same time. One travels north at 25 mph, and the other travels east at 60 mph. How fast is the distance between them increasing at the end of 1 hour? [Hint: $D^2 = x^2 + y^2$. To find D after 1 hour, solve $D^2 = 25^2 + 60^2$.]



$D^2 = x^2 + y^2 \quad \rightarrow \quad \frac{d}{dt}\left(D^2\right) = \frac{d}{dt}\left(x^2 + y^2\right) \quad \rightarrow \quad 2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \quad \rightarrow$

$\frac{dD}{dt} = \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2D} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{D}$

From the given information, we know that after 1 hour x = 25 miles, y = 60 miles, and

$D = \sqrt{25^2 + 60^2} = \sqrt{625 + 3600} = \sqrt{4225} = 65 \text{ miles}$

$\frac{dx}{dt} = 25 \text{ mph} \qquad \frac{dy}{dt} = 60 \text{ mph}$

Therefore, $\frac{dD}{dt} = \frac{25(25) + 60(60)}{65} = \frac{625 + 3600}{65} = \frac{4225}{65} = 65 \text{ mph}$