3.1 Exponential Functions

I. Review of the Laws of Exponents

$a^x \cdot a^y = a^{x+y}$ $\quad \dfrac{a^x}{a^y} = a^{x-y}$ $\quad (a^x)^y = a^{x \cdot y}$ $\quad a^{-x} = \dfrac{1}{a^x}$ $\quad a^0 = 1$

$(a \cdot b)^x = a^x \cdot b^x$ $\quad \left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$ $\quad \left(\dfrac{a}{b}\right)^{-x} = \left(\dfrac{b}{a}\right)^x$ $\quad a^{\frac{x}{y}} = \sqrt[y]{a^x}$

II. Definition of an Exponential Function

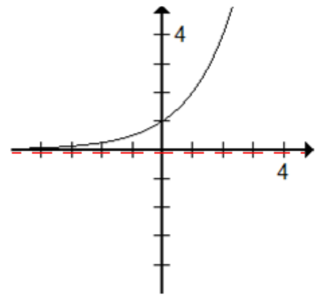
An **exponential function** f is given by $f(x) = a^x$, where x is any real number, a > 0, and $a \neq 1$. The number *a* is called the **base**.

Exponential functions always have a variable in the exponent, not in the base.

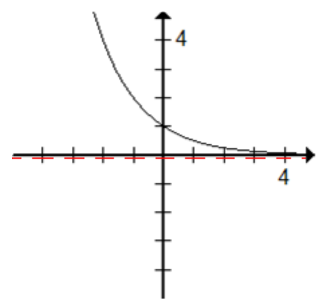
Examples: $f(x) = 3^{1-x}$ $\quad g(x) = \sqrt{2}^x - 4$ $\quad h(x) = \left(\frac{1}{5}\right)^{2x}$

III. Characteristics of the Graphs of Exponential Functions

A. The function given by $f(x) = a^x$, with a > 1, is a positive, increasing, continuous function. It has no minima or maxima, and no inflection points. The graph is concave up and the x-axis is the horizontal asymptote. As x gets smaller, a^x approaches 0 and as x gets larger, a^x approaches positive infinity, i.e. $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = \infty$.



B. The function given by $f(x) = a^x$, with 0 < a < 1, is a positive decreasing, continuous function. It has no minima or maxima, and no inflection points. The graph is concave up and the x-axis is the horizontal asymptote. As x gets larger, a^x approaches 0 and as x gets smaller, a^x approaches positive infinity, i.e. $\lim_{x \to \infty} f(x) = 0$ and $\lim_{x \to -\infty} f(x) = \infty$.



C. $f(x) = \left(\dfrac{1}{a}\right)^x = a^{-x}$

D. The graphs of $f(x) = a^x$ and $g(x) = a^{-x}$ are symmetric with respect to the y-axis.

IV. Graphing an Exponential Function

To graph an exponential function, set the exponent = 0 and solve for x. Make that x-value the center of your t-chart. Plug in two x-values to the left and two x-values to the right. Dot in your horizontal asymptote and connect your 5 points with a smooth curve. State whether the graph is increasing or decreasing and determine its concavity.

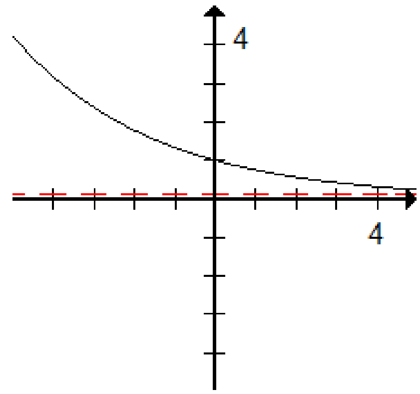
Example 1: Graph $g(x)=\left(\frac{3}{4}\right)^{x}$.

x = 0

Decreasing w/o bound

Concave up

x	y
–2	$\frac{16}{9}$
–1	$\frac{4}{3}$
0	1
1	$\frac{3}{4}$
2	$\frac{9}{16}$



V. The Number *e* and the Derivative of e^x

A. The Definition of *e*

$$e=\lim_{h\to 0}(1+h)^{\frac{1}{h}}\approx 2.718281828459$$

We call *e* the *natural base* or the *natural number*. It was named in honor of Leonhard Euler, the great Swiss mathematician (1707 – 1783) who did groundbreaking work with it.

B. Theorem 1: The Derivative of e^x

The derivative of $f(x)=e^{x}$ is e^{x}.

$$\frac{d}{dx}\left(e^{x}\right)=e^{x}$$

The slope of a tangent line to the graph of $y=e^{x}$ is the same as the function value at x.

Caution: **Notice that we do not take the derivative of e^x by the Power Rule. This is because the Power Rule applies to x^n, a variable raised to a constant power, whereas e^x is a constant raised to a variable power.**

C. Theorem 2: The Derivative of $e^{f(x)}$

The derivative of e to some power is the product of e to that power and the derivative of the power.

$$\frac{d}{dx}\left(e^{f(x)}\right)=e^{f(x)}\cdot f'(x) \quad \text{or} \quad \frac{d}{dx}e^{u}=e^{u}\cdot\frac{du}{dx}$$

Example 2: Differentiate $G(x)=x^{3}-5e^{2x}$.

$$G'(x)=3x^{2}-5(e^{2x}\cdot 2)=3x^{2}-10e^{2x}$$

Example 3: Differentiate $f(x)=x^{7}\cdot e^{4x}$.

$$f'(x)=7x^{6}\cdot e^{4x}+e^{4x}\cdot 4\cdot x^{7}=7x^{6}e^{4x}+4x^{7}e^{4x}$$

Example 4: Differentiate $y=e^{x}+x^{3}-x\cdot e^{x}$.

$$y'=e^{x}+3x^{2}-(1\cdot e^{x}+e^{x}\cdot x)=e^{x}+3x^{2}-e^{x}-xe^{x}=3x^{2}-xe^{x}$$

Example 5: Differentiate $g(x)=\left(4x^2+3x\right)\cdot e^{x^2-7x}$.

$$g'(x)=(8x+3)\cdot e^{x^2-7x}+e^{x^2-7x}\cdot(2x-7)\cdot\left(4x^2+3x\right)$$

D. <u>Characteristics of the Graph of f(x) = e^x</u>

The graph of $f(x)=e^x$ is a positive, increasing, continuous function. It has no critical values, no maximum or minimum values, and no points of inflection. The graph is concave up with a horizontal asymptote at the x-axis. As x gets smaller, e^x approaches 0 and as x gets larger, e^x approaches positive infinity, i.e. $\lim_{x\to -\infty} f(x)=0$ and $\lim_{x\to\infty} f(x)=\infty$.

Example 6: Graph each function then determine critical values, inflection points, intervals over which the function is increasing or decreasing, and the concavity.

$G(x)=-e^{\frac{1}{2}x}$

$\frac{1}{2}x=0 \;\rightarrow\; x=0$

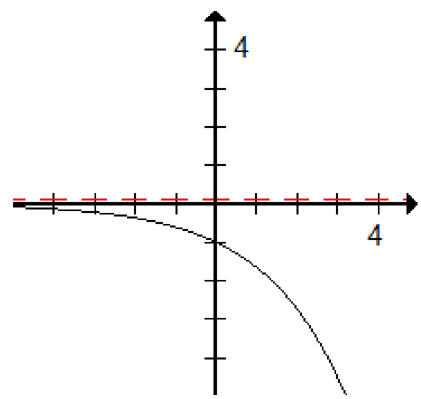
No critical values

No inflection points

Decreasing w/o bound

Concave down

x	y
–2	$-\frac{1}{e}\approx -.368$
–1	$-\frac{1}{e^{\frac{1}{2}}}\approx -.607$
0	–1
1	$-e^{\frac{1}{2}}\approx -1.649$
2	$-e\approx -2.718$



VI. Applications

Exponential functions are used for modeling and solving a wide variety of real world problems involving growth and decay. Examples include calculating the growth of money at compound interest; the growth of populations of people, animals, and bacteria; radioactive decay; marginals; the effects of inflation; and the growth or decline of exports and imports.

Example 7: U.S. exports of goods are increasing exponentially. The value of the exports, t years after 2009, can be approximated by $V(t)=1.6e^{.046t}$ where t = 0 corresponds to 2009 and V is in billions of dollars. (*Source:* U.S. Commerce Department)

a. Estimate the value of U.S. exports in 2009 and 2020.

b. What is the doubling time for the value of U.S. exports?

a. 2009 $\rightarrow$ t = 0   2020 $\rightarrow$ t = 11

$V(0)=1.6e^{(.046\cdot 0)}=1.6$ billion dollars

$V(11)=1.6e^{(.046\cdot 11)}=2.7$ billion dollars

b. doubling time: time to go from initial amount (1.6 billion) to double that (3.2 billion)

$3.2=1.6e^{.046t} \;\rightarrow\; 2=e^{.046t} \;\rightarrow\; \ln 2=\ln e^{.046t} \;\rightarrow\; \ln 2=.046t \;\rightarrow$

$t=\dfrac{\ln 2}{.046}\approx 15$ years