# 3.2 Logarithmic Functions

## I. Introduction

Logarithms were invented by John Napier (1550 – 1617) in the late 16th century as a means of simplifying complex calculations. Their invention enabled astronomer Johannes Kepler (1571 – 1630) to accurately describe, for the first time, the orbits and motions of the planets. Today we use calculators or computers to carry out complex calculations, but logarithms are still used for manipulating exponential models.

## II. Definition of a Logarithm

A **logarithm** is defined as follows:

$\log_a x = y$ means $a^y = x$ for a > 0 and a ≠ 1.

The number $\log_a x$ is the power *y* to which we raise *a* to get *x*. The number *a* is called the **base** of the logarithm. *x* is called the **argument** of the logarithm. We read $\log_a x$ as "log, base *a*, of *x*".

When all is said and done, a logarithm is an exponent or power. The expression $\log_2 8$ means the power of 2 which equals 8. Since $2^3 = 8$, $\log_2 8 = 3$. When you see $\log_a x$, think "*a* to what power equals *x*?"

**Examples:**

$\log_4 64$ → 4 to what power equals 64? → $4^3 = 64$ → Therefore, $\log_4 64 = 3$

$\log_3 \frac{1}{9}$ → 3 to what power equals $\frac{1}{9}$? → $3^{-2} = \frac{1}{9}$ → Therefore, $\log_3 \frac{1}{9} = -2$

## III. Common Logarithms versus Natural Logarithms

A base 10 logarithm, $\log_{10} x$, is called a **common logarithm**. It is frequently written as just *log x*, so if we see a logarithm without a base, we assume it is a base 10 logarithm.

A base e logarithm, $\log_e x$, is called a **natural logarithm**. To avoid confusion with the common log, it is written as *ln x*.

## IV. Converting Between Logarithmic and Exponential Form

The logarithmic equation $\log_b A = P$ is equivalent to the exponential equation $b^P = A$.

A. To convert from log form to exponential form:

Take the base of the logarithm (b) raise it to the power on the other side of the equal sign (P) and set that equal to the argument (A).

**Example 1:** Write an equivalent exponential equation.

$\log_a J = K$ → $a^K = J$

B. To convert from exponential form to log form:

Make the base of the exponent (b) the base of the logarithm. Use the number on the other side of the equal sign (A) as the argument and set the log equal to the power (P).

**Example 2:** Write an equivalent logarithmic equation.

$Q^n = T$ → $\log_Q T = n$

## V. Basic Properties of Logarithms (Theorem 3)

For any positive numbers M, N, a, and b, with a, b ≠ 1, and any real number k:

P1. $\log_a(M \cdot N) = \log_a M + \log_a N$

P2. $\log_a \frac{M}{N} = \log_a M - \log_a N$

P3. $\log_a(M^k) = k \cdot \log_a M$

P4. $\log_a a = 1$

P5. $\log_a(a^k) = k$

P6. $\log_a 1 = 0$

P7. $\log_b M = \frac{\log_a M}{\log_a b}$ (The change of base formula.)

Note: These properties are true for logarithms with any base, including base 10 logarithms (log) and base e logarithms (ln).

**Example 3:** Given $\log_b 3 = 1.099$ and $\log_b 5 = 1.609$, find $\log_b 15$. Do not use a calculator.

$\log_b 15 = \log_b(3 \cdot 5) = \log_b 3 + \log_b 5$ =1.099 + 1.609 = 2.708

**Example 4:** Given $\ln 4 = 1.3863$ and $\ln 5 = 1.6094$, find $\ln \frac{1}{4}$. Do not use a calculator.

$\ln \frac{1}{4} = \ln 1 - \ln 4 = 0 - 1.3863 = -1.3863$

## VI. Exponential Equation

Exponential functions and logarithmic functions are inverses which undo one another. To solve an exponential equation, we must use a logarithm to isolate the variable in the exponent.

**Example 5:** Solve for t.

$$e^{3t} = 900$$
$$\ln e^{3t} = \ln 900$$
$$3t = \ln 900$$
$$t = \frac{\ln 900}{3} \approx 2.267$$

## VII. Derivatives of Natural Logarithmic Functions

A. The Derivative of the Natural Logarithm (Theorem 6)

For any positive number x, $\frac{d}{dx} \ln x = \frac{1}{x}$, for x > 0.

B. The Derivative of the Natural Logarithm of a Function (Theorem 7)

The derivative of the natural logarithm of a function is the derivative of the function divided by the function:

$$\frac{d}{dx} \ln f(x) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)} \quad \text{or} \quad \frac{d}{dx} \ln u = \frac{1}{u} \cdot \frac{du}{dx}$$

**Example 6:** Differentiate $y = -8\ln x$.

$$y' = -8\left(\frac{1}{x}\right) = -\frac{8}{x}$$

**Example 7:** Differentiate $y = x^4 \cdot \ln x - \frac{1}{2}x^2$.

$$y' = 4x^3 \cdot \ln x + \frac{1}{x} \cdot x^4 - x = 4x^3 \cdot \ln x + x^3 - x$$

**Example 8:** Differentiate $y = \frac{\ln x}{x^4}$.

$$y' = \frac{x^4 \cdot \frac{1}{x} - \ln x \cdot 4x^3}{\left(x^4\right)^2} = \frac{x^3 - 4x^3 \cdot \ln x}{x^8} = \frac{x^3(1 - 4\ln x)}{x^3 \cdot x^5} = \frac{1 - 4\ln x}{x^5}$$

**Example 9:** Differentiate $y = \ln \frac{x^4}{2}$.

$$y = \ln \frac{x^4}{2} = \ln \frac{1}{2}x^4$$

$$y' = \frac{\frac{1}{2} \cdot 4x^3}{\frac{1}{2}x^4} = \frac{4}{x}$$

**Example 10:** Differentiate $f(x) = \ln\left(\frac{x^2 + 5}{x}\right)$.

$$f(x) = \ln\left(\frac{x^2 + 5}{x}\right) = \ln\left[x^{-1} \cdot \left(x^2 + 5\right)\right]$$

$$f'(x) = \frac{-x^{-2} \cdot \left(x^2 + 5\right) + 2x \cdot x^{-1}}{x^{-1} \cdot \left(x^2 + 5\right)} = \frac{-1 - 5x^{-2} + 2}{x + 5x^{-1}} = \frac{1 - 5x^{-2}}{x + 5x^{-1}}$$

$$= \frac{\left(1 - 5x^{-2}\right)}{\left(x + 5x^{-1}\right)} \cdot \frac{x^2}{x^2} = \frac{x^2 - 5}{x^3 + 5x}$$

**Example 11:** Differentiate $g(x) = (\ln x)^3$.

$$g'(x) = 3(\ln x)^2 \cdot \frac{1}{x} = \frac{3(\ln x)^2}{x}$$

## VIII. The Graph of f(x) = ln x (Theorem 5)

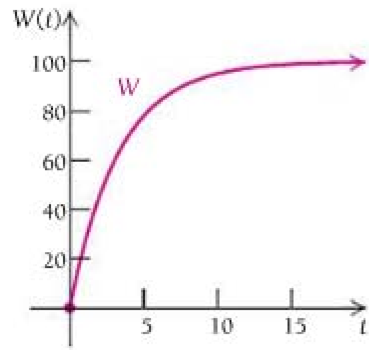
The graph of f(x) is an increasing function with no critical values, no maximum or minimum values, and no points of inflection. The domain is (0, ∞). The range is $(-\infty, \infty)$. The graph is concave down with $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to 0} f(x) = -\infty$.

## IX. Applications

There are numerous real-world applications of logarithmic functions. Examples include calculating the response of consumers to advertising, the growth of stocks, marginals, forgetting time, learning curves, and walking speed.

**Example 12:** Hullian Learning Model

A keyboarder learns to type *W* words per minute after *t* weeks of practice, where *W* is given by $W(t) = 100\left(1 - e^{-.3t}\right)$.




a. Find W(1) and W(8).

b. Find $W'(t)$.

c. After how many weeks will the keyboarder's speed be 95 words per minute?

a. $W(1) = 100\left(1 - e^{-.3(1)}\right) \approx 26$ words per minute

$W(8) = 100\left(1 - e^{-.3(8)}\right) \approx 91$ words per minute

b. $W(t) = 100\left(1 - e^{-.3t}\right) = 100 - 100e^{-.3t}$

$W'(t) = -100e^{-.3t} \cdot -.3 = 30e^{-.3t}$

c. $100\left(1 - e^{-.3t}\right) = 95$

$1 - e^{-.3t} = .95$

$-e^{-.3t} = -.05$

$e^{-.3t} = .05$

$\ln e^{-.3t} = \ln .05$

$-.3t = \ln .05$

$t = \dfrac{\ln .05}{-.3} \approx 10$ weeks