4.1 Antidifferentiation

I. Antiderivatives and Indefinite Integrals

A. Introduction

Antidifferentiation (integration) is used to recover an original function *f* from its derivative f'. For example, the antiderivative of $3x^2$ is x^3 since the derivative of x^3 is $3x^2$. However, $3x^2$ is also the derivative of $x^3 + 1$ and $x^3 - 2$. In fact, $3x^2$ is the derivative of any function in the form $x^3 + C$ so we will use this more general "plus C form" to represent the antiderivative of $3x^2$.

B. Definition of an Antiderivative (Theorem 1)

The **antiderivative** of f(x) is the set of functions F(x) + C such that $\frac{d}{dx}[F(x)+C]=f(x)$.

The constant C is called the **constant of integration**.

Theorem 1 can be restated as follows: if two functions F(x) and G(x) have the same derivative f(x), then F(x) and G(x) differ at most by a constant C, F(x) = G(x) + C.

C. Symbols and Terminology

If F(x) is an antiderivative of a function f(x), we write $\int f(x)\,dx = F(x)+C$. The expression on the left side is called an **indefinite integral**. The symbol $\int$ is the **integral sign** and is a command for antidifferentiation. The function f(x) is the **integrand**. dx indicates that the variable of integration is x. C is considered an arbitrary constant since it can be negative, zero, or positive.

Note: Every integral must contain an appropriate indicator of the variable of integration.

D. Definition of an Indefinite Integral

$\int f(x)\,dx = F(x)+C$ if and only if $F'(x)=f(x)$.

In words, the integral of f(x) is F(x) + C if and only if the derivative of F(x) is f(x).

An indefinite integral can always be checked by differentiation. The derivative of the answer must equal the integrand.

II. Rules of Antidifferentiation (Theorem 2)

A1. Constant Rule

$\int k\,dx = kx + C$ The integral of a constant is the constant times x plus C.

Hint: The integral of 1 is x plus a constant. $\int 1\,dx = x + C$.

A2. Power Rule (where $n \neq -1$)

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

To integrate x to a power other than –1, add 1 to the exponent and divide by the new exponent.

Hint: You may need to rewrite or expand some integrands in exponential form before you can integrate.

A3. Natural Logarithm Rule

$$\int \frac{1}{x}\,dx = \ln x + C$$

A4. Exponential Rule (base e)

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$$

III. Properties of Antidifferentiation (Theorem 3)

P1. $\int [c \cdot f(x)]\,dx = c \cdot \int f(x)\,dx$

A constant factor can be moved to the front of an indefinite integral.

P2. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

The antiderivative of a sum or difference is the sum or difference of the antiderivatives.

Example 1 Determine the indefinite integral. $\int (3t^2 - 4t + 7)\,dt$

$$\int (3t^2 - 4t + 7)\,dt = 3\left(\frac{t^{2+1}}{2+1}\right) - 4\left(\frac{t^{1+1}}{1+1}\right) + 7t + C = 3\left(\frac{t^3}{3}\right) - 4\left(\frac{t^2}{2}\right) + 7t + C = t^3 - 2t^2 + 7t + C$$

Note: As soon as you take the antiderivative, the integral symbol disappears and + C appears.

Example 2 Determine the indefinite integral. $\int \frac{-7}{\sqrt[3]{x^2}}\,dx$

$$\int \frac{-7}{\sqrt[3]{x^2}}\,dx = \int -7x^{-\frac{2}{3}}\,dx = -7\,\frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + C = -7\left(\frac{x^{\frac{1}{3}}}{\frac{1}{3}}\right) = -21x^{\frac{1}{3}} + C$$

Example 3 Determine the indefinite integral. $\int (5x^2 - 2e^{7x})\,dx$

$$\int (5x^2 - 2e^{7x})\,dx = 5\left(\frac{x^{2+1}}{2+1}\right) - 2\left(\frac{e^{7x}}{7}\right) + C = 5\left(\frac{x^3}{3}\right) - 2\left(\frac{e^{7x}}{7}\right) + C = \frac{5}{3}x^3 - \frac{2}{7}e^{7x} + C$$

Example 4 Determine the indefinite integral. $\int \left(x^4 + \frac{1}{8\sqrt{x}} - \frac{4}{5}x^{-\frac{2}{5}}\right)dx$

$$\int \left(x^4 + \frac{1}{8\sqrt{x}} - \frac{4}{5}x^{-\frac{2}{5}}\right)dx = \int \left(x^4 + \frac{1}{8}x^{-\frac{1}{2}} - \frac{4}{5}x^{-\frac{2}{5}}\right)dx = \frac{x^{4+1}}{4+1} + \frac{1}{8}\,\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \frac{4}{5}\,\frac{x^{-\frac{2}{5}+1}}{-\frac{2}{5}+1} + C$$

$$= \frac{x^5}{5} + \frac{1}{8}\,\frac{x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{4}{5}\,\frac{x^{\frac{3}{5}}}{\frac{3}{5}} + C = \frac{1}{5}x^5 + \frac{1}{8}\cdot\frac{2}{1}x^{\frac{1}{2}} - \frac{4}{5}\cdot\frac{5}{3}x^{\frac{3}{5}} + C = \frac{1}{5}x^5 + \frac{1}{4}x^{\frac{1}{2}} - \frac{4}{3}x^{\frac{3}{5}} + C$$

Example 5 Determine the indefinite integral. $\int \left(\frac{4}{\sqrt[5]{x}} + \frac{3}{4}e^{6x} - \frac{7}{x} \right) dx$

$$\int \left(\frac{4}{\sqrt[5]{x}} + \frac{3}{4}e^{6x} - \frac{7}{x} \right) dx = \int \left(4x^{-\frac{1}{5}} + \frac{3}{4}e^{6x} - \frac{7}{x} \right) dx = 4\left(\frac{x^{-\frac{1}{5}+1}}{-\frac{1}{5}+1} \right) + \frac{3}{4}\left(\frac{e^{6x}}{6} \right) - 7\ln x + C$$

$$= 4\left(\frac{x^{\frac{4}{5}}}{\frac{4}{5}} \right) + \frac{3}{24}e^{6x} - 7\ln x + C = 4 \cdot \frac{5}{4}x^{\frac{4}{5}} + \frac{1}{8}e^{6x} - 7\ln x + C = 5x^{\frac{4}{5}} + \frac{1}{8}e^{6x} - 7\ln x + C$$

IV. Initial Conditions

An **initial condition** is an ordered pair that is a solution of a particular antiderivative of an integrand. It is used to find the value of C, the constant of integration.

Example 6 Find f such that $f'(x) = 6x^2 - 4x + 2,\ f(1) = 9$

$$f(x) = \int \left(6x^2 - 4x + 2\right) dx = 6 \cdot \frac{x^3}{3} - 4 \cdot \frac{x^2}{2} + 2x + C = 2x^3 - 2x^2 + 2x + C$$

$$2(1)^3 - 2(1)^2 + 2(1) + C = 9 \quad \rightarrow \quad 2 + C = 9 \quad \rightarrow \quad C = 7$$

$$f(x) = 2x^3 - 2x^2 + 2x + 7$$

Example 7 Total Cost from Marginal Cost

A company determines that the marginal cost, C', of producing the x^{th} unit of a product is given by $C'(x) = x^3 - x$. Find the total-cost function, C, assuming that C(x) is in dollars and fixed costs are $6500.

$$C(x) = \int \left(x^3 - x\right) dx \quad \rightarrow \quad C(x) = \frac{x^4}{4} - \frac{x^2}{2} + K \quad \rightarrow \quad 6500 = \frac{(0)^4}{4} - \frac{(0)^2}{2} + K \quad \rightarrow$$

$$6500 = 0 + 0 + K \quad \rightarrow \quad K = 6500 \quad \rightarrow \quad C(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + 6500$$

Hint: When working with a cost function, C(x), use K to represent the constant of integration.