4.2 Antiderivatives as Areas

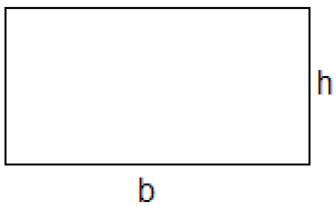
I. Introduction

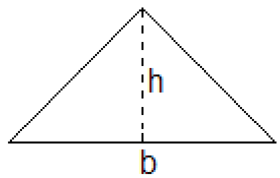
One of the primary applications of integral calculus is calculating the area below the graph of a function (specifically, the area between the graph of the function and the x-axis). This area may represent a wide variety of things. A few that we will look at are total distance traveled by a vehicle, the total cost of producing x units of a product, the total revenue from selling x units of a product, and the total profit from producing and selling x units of a product.

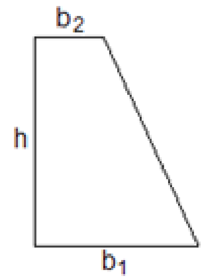
II. Geometry and Areas

In our initial approach to calculating the area formed by the graph of a function, we will use the following geometric formulas:

Area of a rectangle: $A = b \cdot h$ Area of a triangle: $A = \frac{1}{2} b \cdot h$ Area of a trapezoid: $A = \frac{1}{2} h \cdot (b_1 + b_2)$







The units of area are found by multiplying the units of the input variable by the units of the output variable. It is crucial that the units are consistent.

Example 1 **Total cost from marginal cost**

Sylvie's Old World Cheese has found that the cost, in dollars per kilogram, of the cheese it produces is

$C'(x) = -.003x + 4.25$, for $x \le 500$,

where x is the number of kilograms of cheese produced. Find the total cost of producing 400 kg of cheese.

Viewing the trapezoid sideways, we have

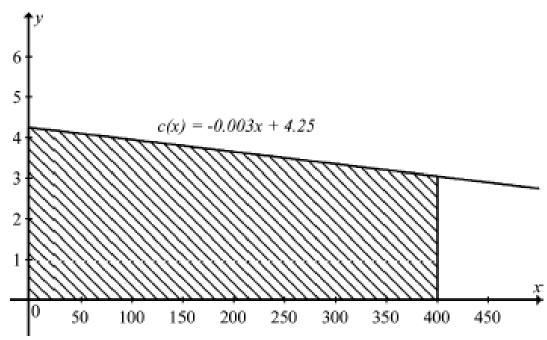
h = 400

$b_1 = C'(0) = -.003(0) + 4.24 = 4.25$

$b_2 = C'(400) = -.003(400) + 4.25 = 3.05$

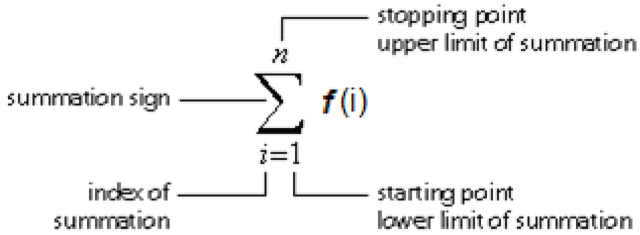
Thus, the total cost of producing 400 kg is

$C = \frac{1}{2} \cdot 400 \cdot (4.25 + 3.05) = \$1460$.



III. Summation Notation

The Greek capital letter sigma, $\sum$ is used in mathematics to mean summation.



If f(i) is a function involving i, the expression $\sum_{i=1}^{n} f(i)$ has the following meaning:

$$\sum_{i=1}^{n} f(i) = f(1) + f(2) + f(3) + \ldots + f(n).$$

The "i =" part under the summation sign tells you which number to plug in first to the given expression. The number *n* on top of the summation sign tells you the last number to plug into the given expression. You always increase by one at each successive step.

Example 2 Express $\sum_{i=0}^{5} (-2)^i$ without using summation notation.

$$\sum_{i=0}^{5} (-2)^i = (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 + (-2)^5 = 1 - 2 + 4 - 8 + 16 - 32 = -21$$

Example 3 Write summation notation for the expression: 5 + 10 + 15 + 20 + 25 + 30 + 35

$$5 + 10 + 15 + 20 + 25 + 30 + 35 = 5(1) + 5(2) + 5(3) + 5(4) + 5(5) + 5(6) + 5(7) = \sum_{i=1}^{7} 5i$$

IV. Riemann Summation

A. <u>Introduction</u>

Riemann Summation is a method that allows us to determine the area under curved graphs. We use rectangles to approximate the area under a curve given by y = f(x), a continuous function, over a closed interval [a, b].

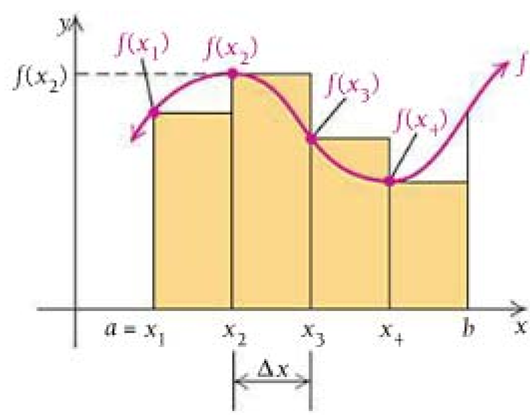
In the adjacent figure, [a, b] is divided into four subintervals, each having width $\Delta x = \frac{b-a}{4}$.

The heights of the rectangles shown are $f(x_1),\ f(x_2),\ f(x_3),\ f(x_4)$.

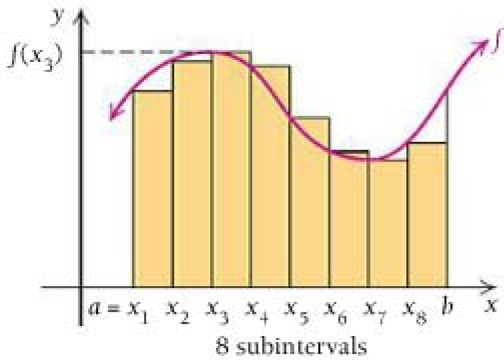
The area of the region under the curve is approximated by the sum of the areas of the four rectangles:

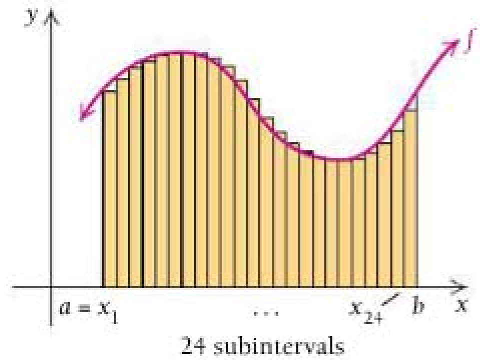
$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x.$$

In sigma notation, the area equals $\sum_{i=1}^{4} f(x_i)\Delta x$.



Approximation of area by rectangles becomes more accurate as we use smaller subintervals and hence more rectangles, as shown in the following figures.





B. <u>Steps for the Process of Riemann Summation</u>

1. Draw the graph of f(x).
2. Subdivide the interval [a, b] into n subintervals of equal width.

   Calculate the width of each rectangle by using the formula $\Delta x = \frac{b-a}{n}$.
3. Construct rectangles above the subintervals such that the top left corner of each rectangle touches the graph.
4. Determine the area of each rectangle by finding $f(x_i)\Delta x$.
5. Sum these areas to arrive at an approximation for the total area under the curve.

Example 4: Total cost from marginal cost

Soulful Scents has found that the cost of producing x ounces of a new fragrance is given by
$C'(x) = .0005x^2 - .1x + 30$, for $x \le 125$,
where $C'(x)$ is in dollars. Use 5 subintervals over [0, 100] and the left endpoint of each subinterval to approximate the cost of producing 100 oz of the fragrance.

$\Delta x = \frac{100-0}{5} = 20$

$x_1 = 0$, $x_2 = 20$, $x_3 = 40$, $x_4 = 60$, $x_5 = 80$

Area I = $C'(0) \cdot 20 = 30 \cdot 20 = 600$

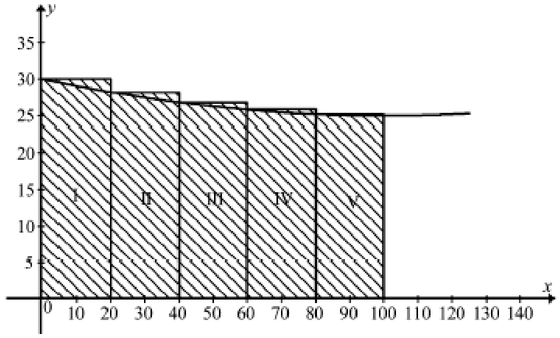
Area II = $C'(20) \cdot 20 = 28.2 \cdot 20 = 564$

Area III = $C'(40) \cdot 20 = 26.8 \cdot 20 = 536$

Area IV = $C'(60) \cdot 20 = 25.8 \cdot 20 = 516$

Area V = $C'(80) \cdot 20 = 25.2 \cdot 20 = 504$

Total cost = 600 + 564 + 536 + 516 + 504 = $2720



Example 5 Approximate the area under the graph of $g(x) = -.02x^4 + .28x^3 - .3x^2 + 20$ over the interval [3, 12] by dividing the interval into 4 subintervals.

$\Delta x = \frac{12-3}{4} = 2.25$

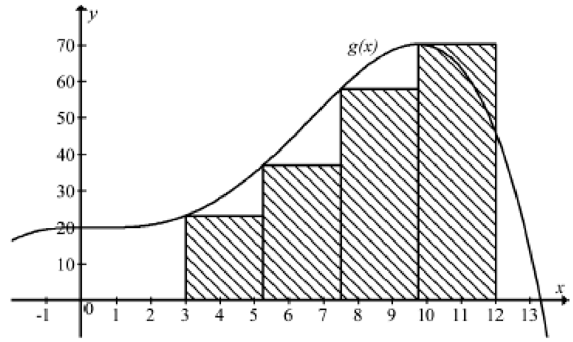
$x_1 = 3$, $x_2 = 5.25$, $x_3 = 7.5$, $x_4 = 9.75$

Area I = $g(3) \cdot 2.25 = 23.24 \cdot 2.25 = 52.29$

Area II = $g(5.25) \cdot 2.25 = 37.054 \cdot 2.25 = 83.372$

Area III = $g(7.5) \cdot 2.25 = 57.969 \cdot 2.25 = 130.43$

Area IV = $g(9.75) \cdot 2.25 = 70.264 \cdot 2.25 = 158.09$



Total area = 52.29 + 83.372 + 130.43 + 158.09 = 424.182

V. Definite Integrals

A. Introduction

The key concept being developed in this section is that the more subintervals we use, the more accurate the approximation of area becomes. As the number of subdivisions *n* increases, the width of each rectangle Δx decreases. If *n* is allowed to approach infinity, then Δx approaches 0; these are limits, and the approximations of area become closer and closer to the true area under the graph. The *exact* area underneath the graph of a continuous function y = f(x) over a closed interval [a, b] is, by definition, given by a definite integral.

B. Definition of a Definite Integral

Let y = f(x) be continuous and nonnegative, f(x) $\geq$ 0, over a closed interval [a, b].
A **definite integral** is the limit as $n \to \infty$ (equivalently, $\Delta x \to 0$) of the Riemann sum of the areas of rectangles under the graph of the function y = f(x) over the interval [a, b].

Exact Area = $\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \int_a^b f(x)\,dx$.

Notice that the summation symbol becomes an integral sign and Δx becomes dx. The interval endpoints *a* and *b* are placed at the bottom and top, respectively, of the integral sign.

Example 6 Use geometry to evaluate the definite integral $\int_2^4 (10 - 2x)\,dx$.

Viewing the trapezoid sideways,

h = 2

$b_1 = f(2) = 10 - 2(2) = 6$

$b_2 = f(4) = 10 - 2(4) = 2$

Area = $\frac{1}{2} \cdot 2 \cdot (6 + 2) = 8$