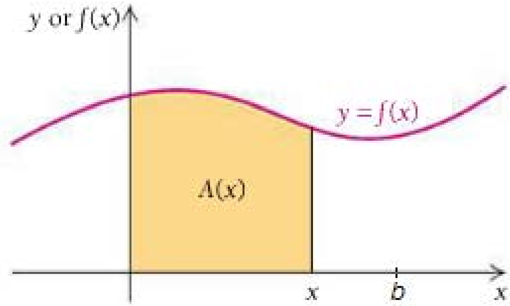
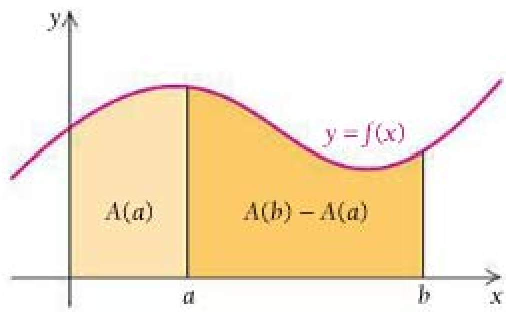
4.3 Area and Definite Integrals

I. The Fundamental Theorem of Calculus (Theorem 4)



Let *f* be a non-negative continuous function over an interval [0, b], and let A(x) be the area between the graph of f and the x-axis over the interval [0, x], with 0 < x < b. Then A(x) is a differentiable function of x and $A'(x) = f(x)$.

This theorem verifies that the derivative of the area function is always the function under which the area is being calculated. However, Theorem 4 only holds for the interval [0, x]. How can we adapt it to apply when *f* is defined over the interval [a, b]?



Referring to the adjacent graph, we see that the area over [a, b] is the same as the area over [0, b] minus the area over [0, a] or A(b) – A (a).

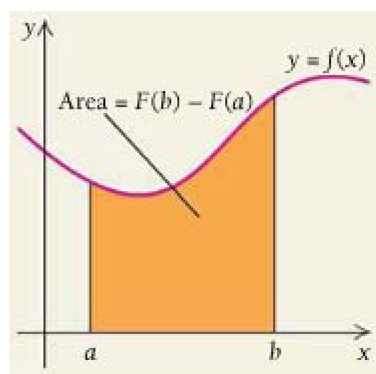
From section 4.1, we know that a function's antiderivatives can differ only in their constant terms.
Thus, if F(x) is another antiderivative of f(x), then A(x) = F(x) + C, for some constant C, and

Area = A(b) – A(a) = F(b) + C – (F(a) + C) = F(b) – F(a).

This result tells us that as long as an area is computed by substituting an interval's endpoints into an antiderivative and then subtracting, *any* antiderivative – and any choice of C – can be used.
It generally simplifies computations to choose 0 as the value of C.

Note: Although it is possible to express the area under a curve as the limit of a Riemann sum, as we did in section 4.2, it is usually much easier to work with antiderivatives.

II. Finding the Area Under a Graph



To find the area under the graph of a non-negative continuous function *f* over the interval [a, b]:

1. Find any antiderivative F(x) of f(x).
   Let C = 0 for simplicity.
2. Evaluate F(x) at x = b and x = a, and compute F(b) – F(a). The result is the area under the graph over the interval [a, b].

Example 1: Find the area under the given curve over the indicated interval.

$y = 1 - x^2$; [–1, 1]

$$F(x) = \int (1 - x^2)\,dx = 1x - \frac{x^3}{3} \qquad [C = 0]$$

$$F(1) - F(-1) = \left(1 - \frac{(1)^3}{3}\right) - \left(-1 - \frac{(-1)^3}{3}\right) = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{2}{3} - \left(-\frac{2}{3}\right) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} = 1\frac{1}{3}$$

Example 2: Find the area under the graph of the function over the given interval.

$y = e^x$, [–2, 3]

$F(x) = \int e^x \, dx = e^x$ [C = 0]

$F(3) - F(-2) = e^3 - e^{-2} \approx 19.950$

III. The Definite Integral

A. Definition of the Definite Integral

Let *f* be any continuous function over the interval [a, b] and F be any antiderivative of *f*. Then the **definite integral** of *f* from a to b is

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

It is convenient to use an intermediate notation : $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$.

Evaluating definite integrals is called *integration*. The numbers a and b are known as the **limits of integration**. Note that this use of the word *limit* indicates an endpoint, not a value that is being approached, as you learned in Chapter 1.

B. Evaluating the Definite Integral

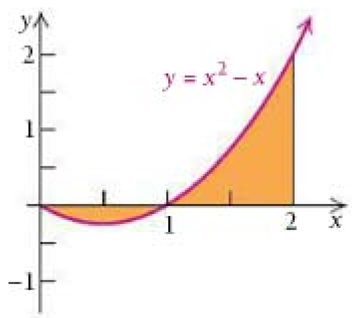
By evaluating the definite integral from a to b, we can find the exact area between the x-axis and the graph of the non-negative continuous function y = f(x) over the interval [a, b].

If a function has areas both below and above the x-axis, the definite integral gives the net total area, or the difference between the sum of the areas above the x-axis and the sum of the areas below the x-axis.

1. If there is more area above the x-axis than below, the definite integral will be positive.
2. If there is more area below the x-axis than above, the definite integral will be negative.
3. If the areas above and below the x-axis are the same, the definite integral will be 0.

Example 3: Evaluate. Then interpret the results in terms of the area above and/or below the x-axis.

$$\int_0^2 (x^2 - x)\,dx$$



$$\int_0^2 (x^2 - x)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^2$$

$$= \left(\frac{2^3}{3} - \frac{2^2}{2}\right) - \left(\frac{0^3}{3} - \frac{0^2}{2}\right)$$

$$= \frac{8}{3} - 2 = \frac{8}{3} - \frac{6}{3} = \frac{2}{3}$$

The area above the x-axis is greater than the area below the x-axis.

IV. The Fundamental Theorem of Integral Calculus

If a continuous function *f* has an antiderivative *F* over [a, b], then

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx = F(b) - F(a)$$

This theorem indicates that we can express the integral of a function either as a limit of a sum or in terms of an antiderivative.

Example 4: Evaluate $\int_{-2}^{5}\left(2x^2 - 3x + 7\right)dx$.

$$\int_{-2}^{5}\left(2x^2 - 3x + 7\right)dx = \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 7x\right]_{-2}^{5} = \left(\frac{2(5)^3}{3} - \frac{3(5)^2}{2} + 7(5)\right) - \left(\frac{2(-2)^3}{3} - \frac{3(-2)^2}{2} + 7(-2)\right)$$

$$= \left(\frac{250}{3} - \frac{75}{2} + 35\right) - \left(-\frac{16}{3} - 6 - 14\right) = \frac{485}{6} - \left(-\frac{152}{6}\right) = \frac{637}{6}$$

V. Applications Involving Definite Integrals

To find total profit, total revenue, or total cost, we integrate the corresponding marginal function.

Because $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$,

to find the position function, s(t), we integrate the velocity function, v(t); and

to find the velocity function, v(t), we integrate the acceleration function, a(t).

Example 5: **Accumulated Sales**

A company estimates that its sales will grow continuously at a rate given by the function $S'(t) = 20e^t$,

where $S'(t)$ is the rate at which sales are increasing, in dollars per day, on day t.

a. Find the accumulated sales for the first 5 days.

b. Find the sales from the 2nd day through the 5th day. (This is the integral from 1 to 5.)

a. $S = \int_0^5 20e^t\,dt = \left[20e^t\right]_0^5 = 20e^5 - 20e^0 = 20e^5 - 20(1) = \$2948.26$

b. $S = \int_1^5 20e^t\,dt = \left[20e^t\right]_1^5 = 20e^5 - 20e^1 = \$2913.90$

Example 6: Find s(t) given a(t) = –2t + 6, with v(0) = 6 and s(0) = 10.

$$v(t) = \int a(t)\,dt = \int (-2t+6)\,dt = -2\left(\frac{t^2}{2}\right) + 6t + C_1 = -t^2 + 6t + C_1$$

$$6 = -(0)^2 + 6(0) + C_1 \;\rightarrow\; C_1 = 6 \;\rightarrow\; v(t) = -t^2 + 6t + 6$$

$$s(t) = \int v(t)\,dt = \int \left(-t^2 + 6t + 6\right)dt = -\frac{t^3}{3} + 6\left(\frac{t^2}{2}\right) + 6t + C_2 = -\frac{t^3}{3} + 3t^2 + 6t + C_2$$

$$10 = -\frac{(0)^3}{3} + 3(0)^2 + 6(0) + C_2 \;\rightarrow\; 10 = C_2 \;\rightarrow\; s(t) = -\frac{t^3}{3} + 3t^2 + 6t + 10$$