## 4.5 Integration Techniques: Substitution

### I. Introduction

The Chain Rule greatly expanded the range of functions that we could differentiate. In this section we will learn the substitution method of integration, essentially the reverse of the chain rule for derivatives. This will greatly expand the range of functions we can integrate.

### II. Differentials

For the differentiable function, f(x), the *differential* df is the derivative of f times dx: $df = f'(x) \cdot dx$.
Note that df does not mean d times f.

Examples: For $f(x) = x^3 - 4x - 2$, the differential df is $df = (3x^2 - 4)dx$.

For $g(x) = e^{x^2+3x}$, the differential dg is $dg = \left[e^{x^2+3x}(2x+3)\right]dx$.

### III. Three Integration Formulas

A. The Power Rule

$$\int u^r du = \frac{u^{r+1}}{r+1} + C, \quad \text{assuming } r \neq -1$$

This rule is used to integrate a function raised to a power $\times$ the differential of the function.

B. The Exponential Rule (base *e*)

$$\int e^u du = e^u + C$$

This rule is used to integrate *e* raised to a power $\times$ the differential of the exponent.

C. The Natural Logarithm Rule

$$\int \frac{1}{u} du = \ln|x| + C \quad \text{or} \quad \int \frac{1}{u} du = \ln x + C, \quad u > 0$$ (Unless otherwise noted, we will assume *u* > 0.)

This rule is used to integrate one over a function $\times$ the differential of the function or
a function to the negative one power $\times$ the differential of the function.

Note: These are the same rules we learned in section 4.1, but here *u* is a function and *du* is the differential of the function.

### IV. Strategy for Substitution

The following strategy may help in carrying out the procedure of substitution:

1. Decide which rule of integration is appropriate.
   a. If you think it is the Power Rule, let *u* be the base.
   b. If you think it is the Exponential Rule (base e), let *u* be the expression in the exponent.
   c. If you think it is the Natural Logarithm Rule, let *u* be the denominator.
2. Write down *u* and *du*.
3. Inspect the integrand to be sure all factors of the differential are included. In some cases we may need to multiply inside and out by a constant (**never a variable**) in order to get the exact differential we need.

   Hint: If you multiply by *a* on the inside, multiply by its reciprocal, $\frac{1}{a}$, on the outside.

4. Once the integrand is complete, make the substitution and then integrate.
5. Reverse the substitution. If there are limits, use them to evaluate the integral **after** the substitution has been reversed.
6. You may check your answer by differentiating.

**Example 1** Evaluate $\int (x^2 - 7)^6 \, 2x \, dx$.

We will use the Power Rule with $u = x^2 - 7$ and $du = 2x\,dx$.

The integrand contains the entire differential, 2x dx. Therefore

$$\int (x^2-7)^6 \, 2x\,dx \;=\; \int u^6\,du \;=\; \frac{1}{7}u^7 + C \;=\; \frac{1}{7}(x^2-7)^7 + C$$

**Example 2** Evaluate $\int (2t^5 - 3)t^4\,dt$.

We will use the Power Rule with $u = 2t^5 - 3$ and $du = 10t^4\,dt$.

The integrand is missing the 10 of the differential, so we will multiply on the inside by 10 and on the outside by $\frac{1}{10}$.

$$\int (2t^5-3)t^4\,dt \;=\; \frac{1}{10}\int (2t^5-3)10t^4\,dt \;=\; \frac{1}{10}\int u\,du \;=\; \frac{1}{10}\left(\frac{1}{2}u^2\right) + C$$

$$=\; \frac{1}{20}u^2 + C \;=\; \frac{1}{20}(2t^5-3)^2 + C$$

**Example 3** Evaluate $\int \frac{2}{1+2x}\,dx$. Assume $u > 0$ when ln u appears.

We will use the Natural Logarithm Rule with $u = 1 + 2x$ and $du = 2\,dx$.

The integrand contains the entire differential, 2 dx. Therefore

$$\int \frac{2}{1+2x}\,dx \;=\; \int \frac{1}{u}\,du \;=\; \ln u + C \;=\; \ln(1+2x) + C$$

**Example 4** Evaluate $\int x^3 e^{x^4}\,dx$.

We will use the Exponential Rule with $u = x^4$ and $du = 4x^3\,dx$.

The integrand is missing the 4 of the differential, so we will multiply on the inside by 4 and on the outside by $\frac{1}{4}$.

$$\int x^3 e^{x^4}\,dx \;=\; \frac{1}{4}\int 4x^3 e^{x^4}\,dx \;=\; \frac{1}{4}\int e^u\,du \;=\; \frac{1}{4}e^u + C \;=\; \frac{1}{4}e^{x^4} + C$$

**Example 5** Evaluate $\int t\left(t^2-1\right)^5 dt$.

We will use the Power Rule with $u = t^2 - 1$ and $du = 2t\,dt$.

The integrand is missing the 2 of the differential, so we will multiply on the inside by 2 and on the outside by $\frac{1}{2}$.

$$\int t\left(t^2-1\right)^5 dt = \frac{1}{2}\int 2t\left(t^2-1\right)^5 dt = \frac{1}{2}\int u^5 du = \frac{1}{2}\left(\frac{1}{6}u^6\right)+C$$

$$= \frac{1}{12}u^6 + C = \frac{1}{12}\left(t^2-1\right)^6 + C$$

**Example 6** Evaluate $\int \frac{e^t dt}{3+e^t}$. Assume $u > 0$ when ln u appears.

We will use the Natural Logarithm Rule with $u = 3 + e^t$ and $du = e^t\,dt$.

The integrand contains the entire differential, $e^t\,dt$. Therefore

$$\int \frac{e^t dt}{3+e^t} = \int \frac{du}{u} = \ln u + C = \ln\left(3+e^t\right)+C$$

**Example 7** Evaluate $\int 5x\sqrt[4]{1-x^2}\,dx$.

First we must rewrite the function: $\int 5x\sqrt[4]{1-x^2}\,dx \rightarrow \int 5x\left(1-x^2\right)^{\frac{1}{4}} dx$

We will use the Power Rule with $u = 1-x^2$ and $du = -2x\,dx$.

The integrand is missing the –2 of the differential, so we will multiply on the inside by $-\frac{2}{5}$ and on the outside by $-\frac{5}{2}$.

$$\int 5x\left(1-x^2\right)^{\frac{1}{4}} dx = -\frac{5}{2}\int -\frac{2}{5}\left[5x\left(1-x^2\right)^{\frac{1}{4}}\right]dx = -\frac{5}{2}\int -2x\left(1-x^2\right)^{\frac{1}{4}} dx$$

$$= -\frac{5}{2}\int u^{\frac{1}{4}}du = -\frac{5}{2}\left(\frac{u^{\frac{5}{4}}}{\frac{5}{4}}\right)+C = -\frac{5}{2}\cdot\frac{4}{5}u^{\frac{5}{4}}+C = -2u^{\frac{5}{4}}+C = -2\left(1-x^2\right)^{\frac{5}{4}}+C$$

**Example 8** Evaluate $\int_0^1 3x^2 e^{x^3} dx$.

We will use the Exponential Rule (base *e*) with $u = x^3$ and $du = 3x^2 dx$.

The integrand contains the entire differential, $3x^2 dx$. Therefore

$$\int_0^1 3x^2 e^{x^3} dx = \int_0^1 e^u du = \left[e^u\right]_{x=0}^{x=1} = \left[e^{x^3}\right]_0^1 = e^{(1)^3} - e^{(0)^3} = e - 1$$

**Example 9** Evaluate $\int_1^4 \frac{2x+1}{x^2+x-1} dx$.

We will use the Natural Logarithm Rule with $u = x^2 + x - 1$ and $du = (2x + 1) dx$.

The integrand contains the entire differential, $du = (2x + 1) dx$. Therefore

$$\int_1^4 \frac{2x+1}{x^2+x-1} dx = \int_1^4 \frac{du}{u} = \left[\ln u\right]_{x=1}^{x=4} = \left[\ln\left(x^2+x-1\right)\right]_1^4$$

$$= \ln\left(4^2+4-1\right) - \ln\left(1^2+1-1\right) = \ln 19 - \ln 1 = \ln 19 - 0 = \ln 19$$

**Example 10** Evaluate $\int_0^2 \frac{3x^2\, dx}{\left(1+x^3\right)^5}$.

We will use the Power Rule with $u = 1 + x^3$ and $du = 3x^2 dx$.

The integrand contains the entire differential, $3x^2 dx$. Therefore

$$\int_0^2 \frac{3x^2\, dx}{\left(1+x^3\right)^5} = \int_0^2 \frac{du}{u^5} = \int_0^2 u^{-5} du = \left[\frac{u^{-4}}{-4}\right]_{x=0}^{x=2} = \left[-\frac{1}{4u^4}\right]_{x=0}^{x=2}$$

$$= \left[-\frac{1}{4\left(1+x^3\right)^4}\right]_0^2 = -\frac{1}{4\left(1+2^3\right)^4} - \left(-\frac{1}{4\left(1+0^3\right)^4}\right) = -\frac{1}{4(9)^4} + \frac{1}{4(1)^4}$$

$$= -\frac{1}{26244} + \frac{1}{4} = \frac{1640}{6561} \approx .2499618961$$