5.1 An Economics Application: Consumer Surplus and Producer Surplus

I. Introduction

In this section we will think of supply and demand as prices that are functions of quantity: p = S(x) and p = D(x). Such an interpretation is common in economics. We will then use integration to calculate Consumer and Producer Surplus – the benefits to consumers and producers of being able to buy and sell at the market price (i.e., the equilibrium point).

II. Terminology

A. Demand Curve

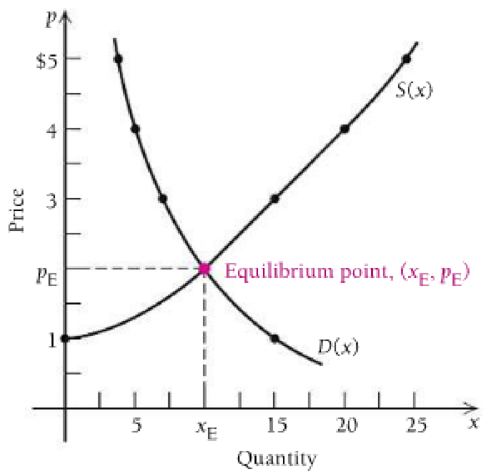
The consumer's **demand curve** is the graph of a function p = D(x), which represents the unit price *p* a consumer is willing to pay for *x* units of a product. It is usually a decreasing function since the consumer expects to pay less per unit for large quantities of the product.

B. Supply Curve

The producer's **supply curve** is the graph of a function p = S(x), which represents the unit price *p* a producer is willing to accept for *x* units of a product. It is usually an increasing function since a higher price per unit is an incentive for the producer to make more units available for sale.

C. Equilibrium Point

The **equilibrium point** (x_E, p_E) is the intersection of the demand and supply curves. It is the point where buyers and sellers come together and purchases and sales actually occur.



D. Utility

The pleasure or benefit a consumer derives from obtaining *x* units of a product is called its **utility**, *U*.

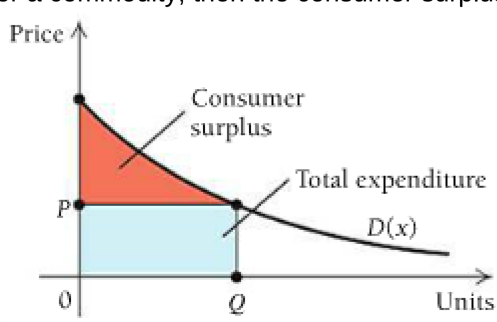
E. Consumer Surplus

Consumer surplus is the extra utility that consumers enjoy when prices decrease as more more units are purchased. It is found by taking the total area under the demand function minus the total expenditure. This is equivalent to the total utility minus the total cost.

If p = D(x) describes the demand function for a commodity, then the consumer surplus is defined for the point (Q, P) as

$$CS = \int_0^Q D(x)\,dx - QP.$$

Note: QP is not included in the integral. It is subtracted after the integral is evaluated.



Example 1 Find the consumer surplus for the demand function given by $D(x) = 840 - .06x^2$ when x = 100.

When x = 100, we have $D(100) = 840 - .06(100)^2 = 240$. Then

$$CS = \int_0^{100} \left(840 - .06x^2\right)dx - 100 \cdot 240 = \left[840x - .02x^3\right]_0^{100} - 24000$$

$$= \left(840(100) - .02(100)^3\right) - \left(840(0) - .02(0)^3\right) - 24000 = (84000 - 20000) - (0 - 0) - 24000$$

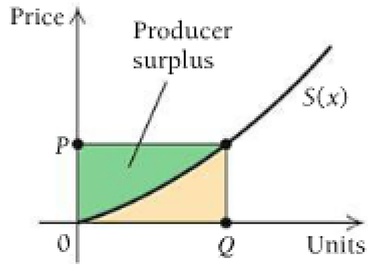
$$= 64000 - 24000 = \$40,000$$

F. Producer Surplus

Producer Surplus is the benefit a producer receives when supplying more units at a price which is higher than he or she expected to receive. It is the extra revenue the producer receives as a result of not being forced to sell fewer units at a lower price. It is found by taking the total receipts minus the area under the supply curve.

If p = S(x) is the supply function for a commodity, then the producer surplus is defined for the point (Q, P) as

$$PS = QP - \int_0^Q S(x)\,dx$$



Example 2 Find the producer surplus for the supply function given by $S(x) = x^2 + 2x + 1$ when x = 10.

When x = 10, we have $S(10) = 10^2 + 2(10) + 1 = 121$. Then

$$PS = 10 \cdot 121 - \int_0^{10} \left(x^2 + 2x + 1\right)dx = 1210 - \int_0^{10} \left(x^2 + 2x + 1\right)dx = 1210 - \left[\frac{1}{3}x^3 + x^2 + x\right]_0^{10}$$

$$= 1210 - \left[\left(\frac{1}{3}(10)^3 + 10^2 + 10\right) - \left(\frac{1}{3}(0)^3 + 0^2 + 0\right)\right] = 1210 - \left[\left(\frac{1000}{3} + 100 + 10\right) - (0)\right]$$

$$= 1210 - \frac{1330}{3} = \$766.67$$

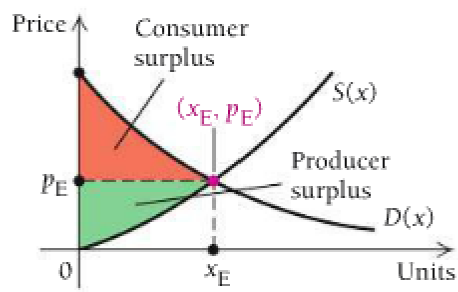
II. Consumer Surplus and Producer Surplus at the Equilibrium Point

Consumers Surplus at the equilibrium point is

$$CS = \int_0^{x_E} D(x)\,dx - x_E \cdot p_E$$

Producer Surplus at the equilibrium point is

$$PS = x_E \cdot p_E - \int_0^{x_E} S(x)\,dx$$



Example 3 D(x) is the price, in dollars per unit, that consumers are willing to pay for x units of an item. S(x) is the price, in dollars per unit, that producers are willing to accept for x units. Find (a) the equilibrium point, (b) the consumer surplus at the equilibrium point, and (c) the producer surplus at the equilibrium point.

$D(x) = (x-4)^2, \quad S(x) = x^2 + 2x + 6$

a. $(x-4)^2 = x^2 + 2x + 6 \quad \to \quad x^2 - 8x + 16 = x^2 + 2x + 6 \quad \to \quad 10 = 10x \quad \to \quad x = 1$

Since $x_E = 1$, $p_E = D(1)$ or $S(1)$ $\to$ $D(1) = (1-4)^2 = (-3)^2 = 9$ $\to$

$(x_E, p_E) = (1, \$9)$

b. $CS = \int_0^1 (x^2 - 8x + 16)dx - 1\cdot 9 = \int_0^1 (x^2 - 8x + 16)dx - 9 = \left[\frac{1}{3}x^3 - 4x^2 + 16x\right]_0^1 - 9$

$= \left[\left(\frac{1}{3} - 4 + 16\right) - (0)\right] - 9 = \frac{37}{3} - 9 = \frac{10}{3} \approx \$3.33$

c. $PS = 1\cdot 9 - \int_0^1 (x^2 + 2x + 6)dx = 9 - \int_0^1 (x^2 + 2x + 6)dx = 9 - \left[\frac{1}{3}x^3 + x^2 + 6x\right]_0^1$

$= 9 - \left[\left(\frac{1}{3}(1)^3 + (1)^2 + 6(1)\right) - (0)\right] = 9 - \left(\frac{1}{3} + 1 + 6\right) = 9 - \frac{22}{3} = \frac{27}{3} - \frac{22}{3} = \frac{5}{3} \approx \$1.67$

Example 4 D(x) is the price, in dollars per unit, that consumers are willing to pay for x units of an item. S(x) is the price, in dollars per unit, that producers are willing to accept for x units. Find (a) the equilibrium point, (b) the consumer surplus at the equilibrium point, and (c) the producer surplus at the equilibrium point.

$D(x) = \frac{1800}{\sqrt{x+1}}, \quad S(x) = 2\sqrt{x+1}$

a. $\frac{1800}{\sqrt{x+1}} = 2\sqrt{x+1} \quad \to \quad \sqrt{x+1}\cdot\frac{1800}{\sqrt{x+1}} = 2\sqrt{x+1}\cdot\sqrt{x+1} \quad \to \quad 1800 = 2(x+1)$

$\to \quad 1800 = 2x + 2 \quad \to \quad 1798 = 2x \quad \to \quad 899 = x$

Since $x_E = 899$, $p_E = D(899)$ or $S(899)$ $\to$

$S(899) = 2\sqrt{899+1} = 2\sqrt{900} = 2(30) = 60$

$(x_E, p_E) = (899, \$60)$

b. $CS = \int_0^{899} \frac{1800}{\sqrt{x+1}}dx - 899(60) = \int_0^{899} \frac{1800}{\sqrt{x+1}}dx - 53940$

$= 1800\int_0^{899} (x+1)^{-\frac{1}{2}}dx - 53940$ Let u = x + 1 and du = 1 dx.

$CS = 1800\int_0^{899} u^{-\frac{1}{2}}du - 53940 = 1800\left[2u^{\frac{1}{2}}\right]_{x=0}^{x=899} - 53940$

$= 1800\left[2(x+1)^{\frac{1}{2}}\right]_0^{899} - 53940 = 1800\left[\left(2(899+1)^{\frac{1}{2}}\right)-\left(2(0+1)^{\frac{1}{2}}\right)\right] - 53940$

$= 1800[60-2]-53940 = 1800(58)-53940 = 104400-53940 = \$50,460$

c. $PS = 899 \cdot 60 - \int_0^{899} 2\sqrt{x+1}\,dx = 53940 - \int_0^{899} 2\sqrt{x+1}\,dx = 53940 - \int_0^{899} 2(x+1)^{\frac{1}{2}}\,dx$

Let u = x + 1 and du = 1 dx

$PS = 53940 - \int_0^{899} 2u^{\frac{1}{2}}\,du = 53940 - \left[2\left(\frac{2}{3}u^{\frac{3}{2}}\right)\right]_{x=0}^{x=899} = 53940 - \left[\frac{4}{3}u^{\frac{3}{2}}\right]_{x=0}^{x=899}$

$= 53940 - \left[\frac{4}{3}(x+1)^{\frac{3}{2}}\right]_0^{899} = 53940 - \left[\left(\frac{4}{3}(899+1)^{\frac{3}{2}} - \frac{4}{3}(0+1)^{\frac{3}{2}}\right)\right]$

$= 53940 - \left[\frac{4}{3}(27000) - \frac{4}{3}\right] = 53940 - \left(36000 - \frac{4}{3}\right) = 53940 - 35998.67$

$= \$17,941.33$