6.1 Functions of Several Variables

I. Introduction

Many of the functions we have looked at thus far have been functions of one variable. For example, the area *A* of a circle is a function of its radius *r*, $A(r) = \pi \cdot r^2$. In contrast, the area *A* of a rectangle is a function of two variables, length *l* and width *w*, $A(l, w) = l \times w$ and the volume *V* of a box is a function of three variables, length *l*, width *w*, and height *h*, $V(l, w, h) = l \times w \times h$.

In this chapter we will look at functions of several variables. We will use the derivatives of multi-variable functions to calculate rates of change and to optimize functions.

II. Functions of Two Variables

A. Definition

A function *f* that depends on two variables, *x* and *y*, is written f(x, y) [read "f of x & y"].

A **function of two variables** is a rule that assigns to each input pair (x, y) exactly one output number z = f(x, y).

B. Domain and Range

The **domain** of f(x, y) is the largest set of all ordered pairs (x, y) [points in the **xy-plane**] for which the function is defined.

The **range** is the set of all resulting z values such that z = f(x, y).

Example 1 Determine the domain of each function of two variables.

a. $f(x, y) = \frac{\sqrt{x}}{\sqrt{y}}$

Since we cannot take an even root of a negative number, $x \geq 0$ and $y \geq 0$.
Since we cannot divide by zero, $y \neq 0$. Therefore, the domain of f(x, y) is
$\{(x, y) \mid x \geq 0 \text{ and } y > 0\}$

b. $f(x, y) = \frac{x}{\ln y}$

Since the numerator contains x by itself, there are no restrictions on x.
Since we cannot take a log of a negative number or of 0, y > 0.
And since ln y is in the denominator, we know ln y cannot equal 0 which means y cannot equal 1. Therefore the domain of f(x, y) is

$\{(x, y) \mid y > 0 \text{ and } y \neq 1\}$ or $\{(x, y) \mid -\infty < x < \infty \text{ and } y > 0 \text{ and } y \neq 1\}$

C. Evaluating a Function in Two Variables

Simply plug in the value given for each variable.

Example 2 Find the following function values.

a. Given $f(x, y) = \sqrt{75 - x^2 - y^2}$, find f(5, –1).

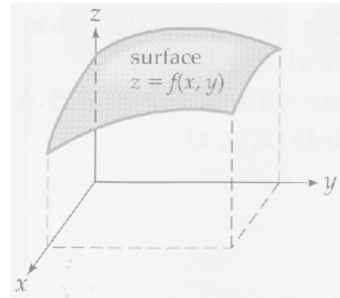
$f(5, -1) = \sqrt{75 - (5)^2 - (-1)^2} = \sqrt{75 - 25 - 1} = \sqrt{49} = 7$

b. Given $g(x, y) = \ln(x^3 - y^2)$, find g(e, 0).

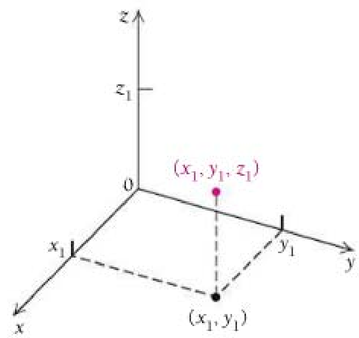
$\ln(e^3 - 0^2) = \ln(e^3) = 3$

III. The Three-Dimensional Coordinate System

The graph of a function of two variables, $z = f(x, y)$ consists of ordered triples (x_1, y_1, z_1), where $z_1 = f(x_1, y_1)$. This graph takes the form of a **surface** above or below the xy-plane. To graph these functions we need a **three-dimensional coordinate system**. Such a system consists of three mutually perpendicular real number lines intersecting at the origin of each line. The x- and y-axes are horizontal and the z-axis is vertical. These axes divide three-dimensional space into 8 octants. In such a system, every ordered triple (x, y, z) is associated with a unique point in space.



To help visualize the three-dimensional coordinate system, look at the left front corner of the room you are sitting in. The bottom corner represents the origin. The horizontal line between the side wall and the floor is the x-axis, the horizontal line between the front wall and the floor is the y-axis, and the vertical line between the side wall and the front wall is the z-axis. The floor is the xy-plane, the side wall is the xz-plane, and the front wall is the yz-plane. To plot the point (x_1, y_1, z_1), locate the point (x_1, y_1) in the xy-plane (on the floor) and move it up or down in space according to the value of z_1.



IV. Applications

Example 3 Price-earnings ratio

The price-earnings ratio of a stock is given by $R(P, E) = \frac{P}{E}$, where P is the price of the stock and E is the earnings per share. The price per share of a miscellaneous stock was \$140, and the earnings per share were \$1.70. Find the price-earnings ratio. Use decimal notation rounded to the nearest hundredth.

$$R(140, 1.70) = \frac{140}{1.70} = 82.35$$

Example 4 Savings and Interest

A sum of \$2500 is deposited in a savings account for which interest is compounded quarterly. The future value A is a function of the annual percentage rate r and the term t, in months, and is given by

$$A(r, t) = 2500\left(1 + \frac{r}{4}\right)^{4t}$$

a. Determine A(.035, 12).

b. What is the interest earned for the rate and term in part a?

c. How much more interest can be earned over the same term as in part a if the APR is increased to 4.25%?

a. $A(.035, 12) = 2500\left(1 + \frac{.035}{4}\right)^{(4\times12)} = \$3{,}797.96$

b. $I_1 = A - P = 3797.96 - 2500 = \$1{,}297.96$

c. $A(.0425, 12) = 2500\left(1 + \frac{.0425}{4}\right)^{(4\times12)} = \$4{,}152.04$

$I_2 = A - P = 4152.04 - 2500 = \$1{,}652.04$

$I_2 - I_1 = 1652.04 - 1297.96 = \$354.08$