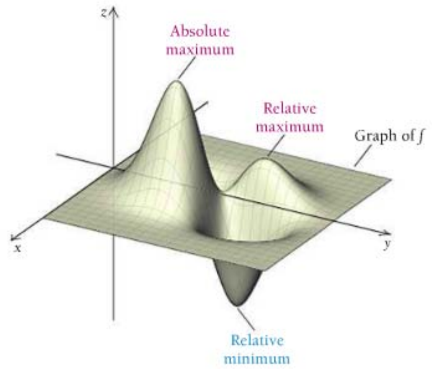
6.3 Maximum – Minimum Problems

I. Relative Extrema of a Function of Two Variables

A. Relative Maximum

A function of two variables, f(x, y) has a **relative maximum** at the point (a, b) if $f(a, b) \geq f(x, y)$ for all points (x, y) in a specified region surrounding (a, b). The value of the relative max is found by plugging the point (a, b) into the f function. Informally, a relative maximum may be thought of as a high point or hilltop of a surface. The **absolute maximum** would be the highest point on the entire domain of the function.

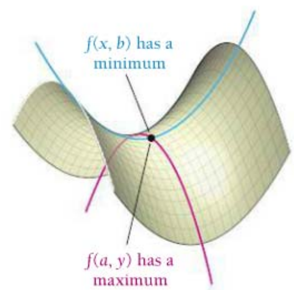


B. Relative Maximum

A function of two variables, f(x, y) has a **relative minimum** at the point (a, b) if $f(a, b) \leq f(x, y)$ for all points (x, y) in a specified region surrounding (a, b). The value of the relative min is found by plugging the point (a, b) into the f function. Informally, a relative minimum may be thought of as a low point or valley bottom of a surface. The **absolute minimum** would be the lowest point on the entire domain of the function.

C. Saddle Point

A **saddle point** is a point that is the highest along one curve of a surface and the lowest along another curve of the same surface. It is like a mountain pass between two peaks. It is not a relative extrema.



II. Critical Points

A point (a, b) is a **critical point** of function f(x, y) if both of its partial derivatives, f_x and f_y, equal zero at that point. To find the critical points for a function f(x, y), set $f_x = 0$ and $f_y = 0$ and solve the resulting system of equations.

Relative maximum and minimum values can occur only at a function's critical points.

III. The D-Test (Theorem 1)

To find the relative maximum and minimum values of a function of two variables, f(x, y):

1. Find f_x, f_y, f_{xx}, f_{xy}, and f_{yy}.
2. Set $f_x = 0$ and $f_y = 0$ and solve the resulting system of equations. Call the solution (a, b). This is a critical point for f.

   Note: If the system has more than one solution, each solution is a critical point, and steps 3 and 4 must be performed on each critical point, one at a time.

3. Find $f_{xx}(a, b)$, $f_{yy}(a, b)$, $f_{xy}(a, b)$, and $D = f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2$.
4. Then,

   a. if D > 0 and $f_{xx}(a, b) < 0$, f has a relative maximum, c, at (a, b). → find c = f(a, b)
   b. if D > 0 and $f_{xx}(a, b) > 0$, f has a relative minimum, c, at (a, b). → find c = f(a, b)
   c. if D < 0, f has a saddle point at (a, b).
   d. if D = 0, the test is inconclusive.

Example 1: Find the relative maximum and minimum values of $f(x,y)=x^2+xy+y^2-y$.

1. $f_x=2x+y$ $\quad f_y=x+2y-1$ $\quad f_{xx}=2$ $\quad f_{xy}=1$ $\quad f_{yy}=2$

2. 2x + y = 0 $\quad$ x + 2y – 1 = 0

   y = –2x $\quad$ x + 2(–2x) – 1 = 0 $\rightarrow$ x – 4x = 1 $\rightarrow$ –3x = 1 $\rightarrow$ $x=-\frac{1}{3}$

   $y=-2\left(-\frac{1}{3}\right)=\frac{2}{3}$ $\quad$ critical point (a, b): = $\left(-\frac{1}{3},-\frac{2}{3}\right)$

3. Since f_{xx}, f_{xy}, and f_{yy} are all constants, we do not have to plug in our critical point to find $f_{xx}(a,b)$, $f_{yy}(a,b)$, and $f_{xy}(a,b)$.

   $D=(2\cdot 2)-(1)^2=4-1=3$

4. Since D > 0 and $f_{xx}(a,b)$ > 0, f has a relative minimum at $f\left(-\frac{1}{3},\frac{2}{3}\right)$.

   $f\left(-\frac{1}{3},\frac{2}{3}\right)=\left(-\frac{1}{3}\right)^2+\left(-\frac{1}{3}\cdot\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2-\frac{2}{3}=-\frac{1}{3}$.

   Conclusion: f has a relative minimum of $-\frac{1}{3}$ at $\left(-\frac{1}{3},\frac{2}{3}\right)$

Example 2: Find the relative maximum and minimum values of $f(x,y)=4xy-x^3-y^2$.

1. $f_x=4y-3x^2$ $\quad f_y=4x-2y$ $\quad f_{xx}=-6x$ $\quad f_{xy}=4$ $\quad f_{yy}=-2$

2. $4y-3x^2=0$ $\quad$ 4x – 2y = 0 $\rightarrow$ 4x = 2y $\rightarrow$ 2x = y

   $4(2x)-3x^2=0$ $\rightarrow$ $8x-3x^2=0$ $\rightarrow$ $x(8-3x)=0$ $\quad$ x = 0

   $8-3x=0$ $\rightarrow$ 8 = 3x $\rightarrow$ $\frac{8}{3}=x$

   $y=2(0)=0$ $\quad y=2\left(\frac{8}{3}\right)=\frac{16}{3}$ $\quad$ critical points: (0, 0) and $\left(\frac{8}{3},\frac{16}{3}\right)$

3. Since f_{xy} and f_{yy} are all constants, we do not have to plug in our critical points to find $f_{xy}(a,b)$ and $f_{yy}(a,b)$. However, we do have to find $f_{xx}(a,b)$ for both critical points.

   For (0, 0), $f_{xx}(a,b)=f_{xx}(0,0)=-6(0)=0$

   For $\left(\frac{8}{3},\frac{16}{3}\right)$, $f_{xx}(a,b)=f_{xx}\left(\frac{8}{3},\frac{16}{3}\right)=-6\left(\frac{8}{3}\right)=-16$

   For (0, 0), $D=(0\cdot -2)-(4)^2=-16$.

   For $\left(\frac{8}{3},\frac{16}{3}\right)$, $D=(-16\cdot -2)-(4)^2=32-16=16$

4. Since D < 0 for (0, 0), there is a saddle point at (0, 0).

   Since D > 0 and $f_{xx}(a,b)<0$ for $\left(\frac{8}{3},\frac{16}{3}\right)$, there is a relative maximum at $\left(\frac{8}{3},\frac{16}{3}\right)$.

   $f\left(\frac{8}{3},\frac{16}{3}\right)=4\cdot\frac{8}{3}\cdot\frac{16}{3}-\left(\frac{8}{3}\right)^3-\left(\frac{16}{3}\right)^2=\frac{256}{27}$

   Conclusion: f has a relative maximum of $\frac{256}{27}$ at $\left(\frac{8}{3},\frac{16}{3}\right)$.

Example 3 Maximizing Profit

A concert promoter produce two kinds of souvenir shirts: one kind sells for $18, and the other sells for $25. The total revenue, in thousands of dollars, from the sale of *x* thousand shirts at $18 each and *y* thousand shirts at $25 each is given by

$R(x, y) = 18x + 25y$

The company determines that the total cost, in thousands of dollars, of producing *x* thousand of the $18 shirt and *y* thousand of the $25 shirt is given by

$C(x, y) = 4x^2 - 6xy + 3y^2 + 20x + 19y - 12$.

How many of each type of shirt must be produced and sold in order to maximize profit? What is the maximum profit?

Since profit = revenue – cost,

$P(x, y) = 18x + 25y - \left(4x^2 - 6xy + 3y^2 + 20x + 19y - 12\right)$

$P(x, y) = 18x + 25y - 4x^2 + 6xy - 3y^2 - 20x - 19y + 12$

$P(x, y) = -4x^2 + 6xy - 3y^2 - 2x + 6y + 12$

1. $P_x = -8x + 6y - 2$ $\quad P_y = 6x - 6y + 6$ $\quad P_{xx} = -8$ $\quad P_{xy} = 6$ $\quad P_{yy} = -6$

2. $-8x + 6y - 2 = 0$ $\quad 6x - 6y + 6 = 0$

   $-8x + 6y = 2$ $\quad 6x - 6y = -6 \quad \rightarrow \quad -6y = -6x - 6 \quad \rightarrow \quad y = x + 1$

   $-8x + 6(x + 1) = 2 \quad \rightarrow \quad -8x + 6x + 6 = 2 \quad \rightarrow \quad -2x = -4 \quad \rightarrow$

   $x = 2$ $\quad y = 2 + 1 = 3$ $\quad$ critical point: (2, 3)

3. Since f_{xx} , f_{xy} , and f_{yy} are all constants, we do not have to plug in our critical point to find $f_{xx}(a, b)$, $f_{yy}(a, b)$, and $f_{xy}(a, b)$.

   $D = (-8 \cdot -6) - (6)^2 = 48 - 36 = 12$

4. Since D > 0 and $P_{xx} < 0$, P has a relative maximum at (2, 3)

   $P(2,3) = -4(2)^2 + 6 \cdot 2 \cdot 3 - 3(3)^2 - 2(2) + 6(3) + 12 = 19$

   Conclusion: A maximum profit of $19,000 will be earned if 2 thousand $18 shirts and 3 thousand $25 shirts are produced and sold.