

6.5 Constrained Optimization

I. Introduction

In section 6.3 we optimized functions of two variables without placing any restrictions on the independent variables x and y other than the assumption that the point (x, y) lies in the domain of f . However, in many practical optimization problems, we must maximize or minimize a function in which the independent variables are subject to certain constraints. For example, a dog lover might want to build the largest possible enclosure for his dogs using only 40 feet of fencing or a manufacturer may wish to minimize production costs during a labor shortage. In this section we will learn a powerful method for finding the relative extrema of a function $f(x, y)$ whose independent variables x and y are subject to one or more constraints in the form $g(x, y) = 0$. This method involves the use of the variable λ (lambda) called a *Lagrange multiplier* in honor of French mathematician Joseph Louis Lagrange (1736 – 1813).

II. The Method of Lagrange Multipliers

To find a maximum or minimum value of an objective function $f(x, y)$ subject to the constraint $g(x, y) = 0$:

1. Identify the objective function $f(x, y)$ which is to be optimized and the constraint. If necessary, rewrite the constraint with zero on the right.
2. Form the new Lagrange function $F(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y)$ by taking the objective function minus lambda times the constraint function.
3. Find the first partial derivatives of F (F_x, F_y, F_λ) and set each equal to zero.
4. Solve the resulting equations ($F_x = 0, F_y = 0, F_\lambda = 0$) to find the critical points.

Hint: To solve this system it is often helpful to:

Solve $F_x = 0$ and $F_y = 0$ for λ .

Set the two expressions for λ equal to each other and solve for one variable.

Substitute the resulting expression into $F_\lambda = 0$ and solve for the variable.

Back substitute to solve for the other variable.

5. Evaluate f at each of the critical points to find the minimum and / or maximum value of f .

Note: This method only finds the critical points, it does not tell whether the function is maximized, minimized, or neither at these critical points. The D-Test cannot be used on constrained problems, so we are making an assumption that the solution to the original problem exists and that it occurs at one of these critical points.

Some constrained optimization problems can be solved without Lagrange multipliers, but the Lagrange method has two advantages:

1. It eliminates the need to solve the constrained equation for one variable.
2. It is symmetric in the variables allowing you to solve for whichever variable is easier to find.

Example 1 Find the minimum value of f subject to the given constraint.

$$f(x, y) = 2x^2 + y^2 - xy; \quad x + y = 8$$

1. Minimize $f(x, y) = 2x^2 + y^2 - xy$ subject to $x + y = 8 \rightarrow x + y - 8 = 0$
2. $F(x, y, \lambda) = 2x^2 + y^2 - xy - \lambda(x + y - 8) = 2x^2 + y^2 - xy - \lambda x - \lambda y + 8\lambda$
3. $F_x = 4x - y - \lambda \quad F_y = 2y - x - \lambda \quad F_\lambda = -x - y + 8$

$$\begin{aligned}
 4. \quad & 4x - y - \lambda = 0 \quad \rightarrow \quad 4x - y = \lambda \\
 & 2y - x - \lambda = 0 \quad \rightarrow \quad 2y - x = \lambda \\
 & -x - y + 8 = 0
 \end{aligned}$$

Since $4x - y = \lambda$ and $2y - x = \lambda$, $4x - y = 2y - x$.

$$4x - y = 2y - x \quad \rightarrow \quad 5x = 3y \quad \rightarrow \quad x = \frac{3}{5}y$$

Subbing into our last equation, we get

$$-\frac{3}{5}y - y + 8 = 0 \quad \rightarrow \quad -\frac{8}{5}y + 8 = 0 \quad \rightarrow \quad -\frac{8}{5}y = -8 \quad \rightarrow \quad -8y = -40 \quad \rightarrow$$

$$y = 5 \quad \text{Therefore, } x = \frac{3}{5}(5) = 3 \quad \text{and our critical point is } (3, 5).$$

$$5. \quad f(3, 5) = 2(3)^2 + (5)^2 - 3 \cdot 5 = 28$$

Conclusion: Minimum $f = 28$ at $(3, 5)$

Example 2: Use the method of Lagrange multipliers to solve the following.

Of all numbers whose sum is 70, find the two that have the maximum product.

$$1. \quad \text{Maximize } f(x, y) = x \cdot y \text{ subject to } x + y = 70 \quad \rightarrow \quad x + y - 70 = 0$$

$$2. \quad F(x, y, \lambda) = x \cdot y - \lambda \cdot (x + y - 70) = x \cdot y - \lambda x - \lambda y + 70$$

$$3. \quad F_x = y - \lambda \quad F_y = x - \lambda \quad F_\lambda = -x - y + 70$$

$$4. \quad y - \lambda = 0 \quad \rightarrow \quad y = \lambda$$

$$x - \lambda = 0 \quad \rightarrow \quad x = \lambda$$

$$-x - y + 70 = 0$$

Since $x = \lambda$ and $y = \lambda$, $x = y$

Subbing into our last equation, we get

$$-x - x + 70 = 0 \quad \rightarrow \quad -2x = -70 \quad \rightarrow \quad x = 35$$

Since $x = 35$, $y = 35$ and our critical point is $(35, 35)$

$$5. \quad f(35, 35) = 35 \cdot 35 = 1225$$

Conclusion: Of all numbers whose sum is 70, 35 & 35 have the maximum product, 1225.

Example 3 Maximizing total sales

The total sales, S , of a one-product firm are given by $S(L, M) = ML - L^2$

where M is the cost of materials and L is the cost of labor.

Find the maximum value of this function subject to the budget constraint $M + L = 90$.

$$1. \quad \text{Maximize } S(L, M) = ML - L^2 \text{ subject to } M + L = 90 \quad \rightarrow \quad M + L - 90 = 0$$

$$2. \quad F(L, M, \lambda) = ML - L^2 - \lambda(M + L - 90) = ML - L^2 - \lambda M - \lambda L + 90$$

$$3. \quad F_L = M - 2L - \lambda \quad F_M = L - \lambda \quad F_\lambda = -M - L + 90$$

$$4. \quad M - 2L - \lambda = 0 \quad \rightarrow \quad M - 2L = \lambda$$

$$L - \lambda = 0 \quad \rightarrow \quad L = \lambda$$

$$-M - L + 90 = 0$$

$$\text{Since } \lambda = M - 2L \text{ and } \lambda = L, M - 2L = L$$

$$M - 2L = L \quad \rightarrow \quad M = 3L$$

$$\text{Subbing into our last equation, } -3L - L + 90 = 0 \quad \rightarrow \quad -4L = -90 \quad \rightarrow$$

$$L = 22.5 \quad \text{Therefore } M = 3(22.5) = 67.5 \text{ and our critical point is } (22.5, 67.5).$$

$$5. \quad S(22.5, 67.5) = 22.5 \times 67.5 - (22.5)^2 = 1012.50$$

Conclusion: Maximum total sales are 1012.50 when $L = 22.5$ and $M = 67.5$.