Section 4.1  Graphs of the Sine and Cosine Functions

I. Periodic Functions

A periodic function is one whose function values repeat in a regular pattern. They are used to describe cyclical phenomena. All six trigonometric functions are periodic.

The smallest increment containing one full repetition of the pattern is called the period of the function.

II. The Graph of the “Plain Vanilla” Sine Function

The graph of the “plain vanilla” sine function, \( y = \sin x \), has the following characteristics:

a. Domain: \((-\infty, \infty)\)  Range: \([-1, 1]\)

b. Amplitude: 1

Note: The amplitude is the height above or the depth below the graph’s horizontal axis. It is half the distance between the maximum and minimum y-values.

c. Period: \(2\pi\);  x-increment: \(\frac{\pi}{2}\)  Note: The x-increment is usually \(\frac{1}{4}\) of the period.

d. 5 key points: \((0, 0)\) and \(\left(\frac{\pi}{2}, 1\right)\) and \(\left(\frac{2\pi}{2}, 0\right)\) and \(\left(\frac{3\pi}{2}, -1\right)\) and \(\left(\frac{4\pi}{2}, 0\right)\).

Note that its y-values start in the “neutral position” with \(y_1 = 0\).

e. Sine is an odd function with origin symmetry. For odd functions, \(f(-x) = -f(x)\), so for sine, \(\sin(-x) = -\sin x\). If the point \((a, b)\) is on the graph of an odd function, the point \((-a, -b)\) is also on the graph.
III. The Graph of the “Plain Vanilla” Cosine Function

The graph of the “plain vanilla” cosine function, \( y = \cos x \), has the following characteristics:

a. Domain: \((-\infty, \infty)\) \quad Range: \([-1, 1]\)

b. Amplitude: 1

c. Period: \(2\pi\); \quad x-increment: \(\frac{\pi}{2}\)

d. 5 key points: \((0, 1)\) \(\left(\frac{\pi}{2}, 0\right)\) \(\left(\frac{2\pi}{2}, -1\right)\) \(\left(\frac{3\pi}{2}, 0\right)\) \(\left(\frac{4\pi}{2}, 1\right)\).

Note that its y-values start in the “up position” with \(y_1 = 1\).

e. Cosine is an even function with y-axis symmetry. For even functions, \(f(-x) = f(x)\), so for cosine, \(\cos(-x) = \cos(x)\). Thus, if the point \((a, b)\) is on the graph of an even function, the point \((-a, b)\) is also on the graph.

IV. Transformations of the “Plain Vanilla” Sine and Cosine Functions

A. Amplitude and x-axis reflection

In the functions \(y = a \sin x\) and \(y = a \cos x\), the \(a\)-value controls two things – the amplitude of the graph and whether there has been an x-axis reflection.

The amplitude of the graph is \(|a|\).

If \(a < 0\), the graph has been reflected over the x-axis.

B. Period and x-increment

In the functions \(y = \sin bx\) and \(y = \cos bx\), the \(b\)-value controls the period, which in turn controls the x-increment.

The period is \(\frac{2\pi}{b}\) and the x-increment is \(\frac{1}{4} \times \text{period} \).
Example 1  Find the amplitude, period, and x-increment for each of the given functions and identify whether there has been an x-axis reflection.

a. \( y = \frac{1}{4} \cos 3x \)

\[ a = \frac{1}{4} \rightarrow \text{amp} = \frac{1}{4} ; \text{there is no x-axis reflection.} \]

\[ b = 3 \rightarrow \text{period} = \frac{2\pi}{3} ; \text{x-increment} = \frac{1}{4} \times \frac{2\pi}{3} = \frac{\pi}{6} \]

b. \( y = -5 \sin \pi x \)

\[ a = -5 \rightarrow \text{amp} = 5 ; \text{there is an x-axis reflection} \]

\[ b = \pi \rightarrow \text{period} = \frac{2\pi}{\pi} = 2 ; \text{x-increment} = \frac{1}{4} \times 2 = \frac{1}{2} \]

Example 2  Graph two full periods of \( y = 3 \sin 2x \). Give a, b, amplitude, whether there is an x-axis reflection, period, x-increment, and 5 key points.(#24)

\[ a = 3 \quad \text{amp} = 3 \quad \text{no x-axis reflection} \]

\[ b = 2 \quad \text{period} = \frac{2\pi}{2} = \pi \quad \text{x-increment} = \frac{\pi}{4} \]

\[
\begin{array}{ccccc}
(0, 0) & \left(\frac{\pi}{4}, 3\right) & \left(\frac{2\pi}{4}, 0\right) & \left(\frac{3\pi}{4}, -3\right) & \left(\frac{4\pi}{4}, 0\right) \\
\end{array}
\]

Example 3  Graph two full periods of \( y = -2 \cos \frac{\pi}{4} \). Give a, b, amplitude, whether there is an x-axis reflection, period, x-increment, and 5 key points.(#24)

\[ a = -2 \quad \text{amp} = 2 \quad \text{There is an x-axis reflection} \]

\[ b = \frac{\pi}{4} \quad \text{period} = \frac{2\pi}{\frac{\pi}{4}} = 2 \times \frac{4}{\pi} = 8 \quad \text{x-increment} = \frac{8}{4} = 2 \]

\[
\begin{array}{cccccc}
(0, -2) & (2, 0) & (4, 2) & (6, 0) & (8, -2) \\
\end{array}
\]
Section 4.2 Translations of the Graphs of the Sine and Cosine Functions

I. Phase Shifts (a.k.a. Horizontal Translations)

To find the phase shift of a function in the form \( y = a \sin (bx - c) \) or \( y = a \cos (bx - c) \), set the argument of the function (what is inside parentheses, after the function name) equal to zero and solve for \( x \):

\[
\frac{c}{b} = \frac{-c}{b} \rightarrow \frac{c}{b} \rightarrow x = \frac{c}{b}.
\]

If \( \frac{c}{b} \) is positive (i.e. you see subtraction in the parentheses), the phase shift is \( \frac{c}{b} \) to the right.

If \( \frac{c}{b} \) is negative (i.e. you see addition in the parentheses), the phase shift is \( \frac{c}{b} \) to the left.

II. Vertical Shifts (a.k.a. Vertical Translations)

In the functions \( y = a \sin (bx - c) + d \) and \( y = a \cos (bx - c) + d \), the \( d \) controls the vertical shift. If \( d \) is positive, vertical the shift is \( d \) up. If \( d \) is negative, the vertical shift is \( d \) down.

Example 1 Find the amplitude, period, x-increment, phase shift, and vertical shift for each of the given functions and identify whether there has been an x-axis reflection.

a. \( y = \frac{1}{2} \cos (2x - 3\pi) - 1 \) (#22)  
   \[ a = \frac{1}{2} \rightarrow \text{amp} = \frac{1}{2}; \text{ there is no x-axis reflection.} \]
   \[ b = 2 \rightarrow \text{period} = \frac{2\pi}{2} = \pi; \text{ x-increment} = \frac{1}{4} \times \pi = \frac{\pi}{4} \]
   \[ c = 3\pi \rightarrow 2x - 3\pi = 0 \rightarrow 2x = 3\pi \rightarrow x = \frac{3\pi}{2} \rightarrow \text{phase shift: } \frac{3\pi}{2} \text{ right} \]
   \[ d = -1 \rightarrow \text{vertical shift: } 1 \text{ down} \]

b. \( y = 2 - 3\sin \left( \frac{3}{4}x + \frac{\pi}{8} \right) \)
   \[ a = -3 \rightarrow \text{amp} = 3; \text{ there is an x-axis reflection} \]
   \[ b = \frac{3}{4} \rightarrow \text{period} = \frac{2\pi}{\frac{3}{4}} = \frac{8\pi}{3}; \text{ x-increment} = \frac{1}{4} \times \frac{8\pi}{3} = \frac{2\pi}{3} \]
   \[ c = \frac{\pi}{8} \rightarrow \frac{3}{4}x + \frac{\pi}{8} = 0 \rightarrow \frac{3}{4}x = -\frac{\pi}{8} \rightarrow x = -\frac{\pi}{8} \times \frac{4}{3} = -\frac{\pi}{6} \rightarrow \text{phase shift: } \frac{\pi}{6} \text{ left} \]
   \[ d = 2 \rightarrow \text{vertical shift: } 2 \text{ up} \]

III. How to Graph \( y = a \sin (bx - c) + d \) and \( y = a \cos (bx - c) + d \)

1. Identify a, b, c, and d.
2a. Find the amplitude: \( \text{amp} = |a| \).
   b. Determine if there has been an x-axis reflection: if \( a \) is negative, there has been an x-axis reflection.
3a. Find the period: \( \text{period} = \frac{2\pi}{b} \).
b. Find the x-increment: \( \text{x-increment} = \frac{1}{4} \times \text{period} \).

4. Find the phase shift by setting the argument, \( bx - c \), equal to zero and solving for \( x \):
   
   \[
   \text{P.S.} = \frac{c}{b} \right ( \text{if we see } bx - c \text{ and } \frac{c}{b} \text{ is positive} ) \text{ or } \frac{c}{b} \left ( \text{if we see } bx + c \text{ and } \frac{c}{b} \text{ is negative} \right )
   \]

5. Find the vertical shift: \( |d| \text{ up if } d \text{ is positive or } |d| \text{ down if } d \text{ is negative} \).

6. Identify the five key points:
   
   for sine: \( \left( \frac{c}{b}, d \right), (x_1 + \text{x-inc}, d + a), (x_2 + \text{x-inc}, d), (x_3 + \text{x-inc}, d - a), (x_4 + \text{x-inc}, d) \)

   for cosine: \( \left( \frac{c}{b}, d + a \right), (x_1 + \text{x-inc}, d), (x_2 + \text{x-inc}, d - a), (x_3 + \text{x-inc}, d), (x_4 + \text{x-inc}, d + a) \)

7. Plot the five key points and connect them with a smooth, continuous curve, then extend the graph to two full periods.

**Example 2:** \( y = -3 \sin (2x - \pi) + 1 \)

1. \( a = -3 \) \( b = 2 \) \( c = \pi \) \( d = 1 \)
2. \( \text{amp} = |-3| = 3 \) \( \text{There is an x-axis reflection} \).
3. \( \text{period} = \frac{2\pi}{2} = \pi \) \( \text{x-increment} = \frac{1}{4} \times \pi = \frac{\pi}{4} \)
4. \( \text{phase shift} = \frac{\pi}{2} \text{ or } \frac{2\pi}{4} \right ( \text{right} ) \)
   
   \( 2x - \pi = 0 \rightarrow 2x = \pi \rightarrow x = \frac{\pi}{2} = \frac{2\pi}{4} \)
5. \( \text{vertical shift} = 1 \text{ up} \)
6. 5 key points: \( \left( \frac{2\pi}{4}, 1 \right), \left( \frac{3\pi}{4}, -2 \right), \left( \frac{4\pi}{4}, 1 \right), \left( \frac{5\pi}{4}, 4 \right), \left( \frac{6\pi}{4}, 1 \right) \)

7.
Example 3: \( y = \frac{1}{3} \cos (4x + \pi) - 2 \)

1. \( a = \frac{1}{3} \quad b = 4 \quad c = -\pi \quad d = -2 = -\frac{6}{3} \)

2. \( \text{amp} = \left| \frac{1}{3} \right| = \frac{1}{3} \quad \text{There is no x-axis reflection.} \)

3. \( \text{period} = \frac{2\pi}{4} = \frac{\pi}{2} \quad \text{x-increment} = \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{8} \)

4. \( \text{phase shift} = \frac{\pi}{4} \text{ or } \frac{2\pi}{8} \text{ left} \quad 4x + \pi = 0 \rightarrow 4x = -\pi \rightarrow x = -\frac{\pi}{4} = -\frac{2\pi}{8} \)

5. \( \text{vertical shift} = 2 \text{ or } \frac{6}{3} \text{ down} \)

6. 5 key points: \( \left( -\frac{2\pi}{8}, -\frac{5}{3} \right), \left( -\frac{\pi}{8}, -\frac{6}{3} \right), \left( 0, -\frac{7}{3} \right), \left( \frac{\pi}{8}, -\frac{6}{3} \right), \left( \frac{2\pi}{8}, -\frac{5}{3} \right) \)
Section 4.3  Graphs of the Other Circular Functions

Note: The concept of amplitude does not apply to cosecant, secant, tangent, or cotangent. We can only discuss the amplitude of a sine or cosine function.

I. The Basic Cosecant Function

The graph of \( y = \csc x \) has a period of \( 2\pi \) and an x- increment of \( \frac{\pi}{2} \). It is an odd function with origin symmetry. Because \( \csc x = \frac{1}{\sin x} \), the cosecant function is undefined whenever \( \sin x = 0 \).

Thus the domain of \( \csc x \) is all real numbers except integer multiples of \( \pi \). At \( x = n\pi \), where \( n \) is an integer, the graph of \( y = \sin x \) crosses the horizontal axis and the graph of \( y = \csc x \) has a vertical asymptote. The range of \( \csc x \) is \((-\infty, 1] \cup [1, \infty)\).

The general procedure for graphing \( y = a \csc (bx - c) + d \) is to graph \( y = a \sin (bx - c) + d \) with a dotted line and then to use that graph as a guide for graphing the cosecant. The graph of \( y = \csc x \) has vertical asymptotes at \( x = 0, x = \pi, \text{ and } x = 2\pi \). It has a minimum (low point) at \( \left( \frac{\pi}{2}, 1 \right) \) and a maximum (high point) at \( \left( \frac{3\pi}{2}, -1 \right) \).

II. The Basic Secant Function

The graph of \( y = \sec x \) has a period of \( 2\pi \) and an x- increment of \( \frac{\pi}{2} \). It is an even function with y-axis symmetry. Because \( \sec x = \frac{1}{\cos x} \), the secant function is undefined whenever \( \cos x = 0 \).

Thus the domain of \( \sec x \) is all real numbers except odd multiples of \( \frac{\pi}{2} \). At \( x = \frac{\pi}{2} + n \cdot \pi \), where \( n \) is an integer, the graph of \( y = \cos x \) crosses the horizontal axis and the graph of \( y = \sec x \) has a vertical asymptote. The range of \( \sec x \) is \((-\infty, 1] \cup [1, \infty)\).

The general procedure for graphing \( y = a \sec (bx - c) + d \) is to graph \( y = a \cos (bx - c) + d \) with a dotted line and then to use that graph as a guide for graphing the secant. The graph of \( y = \sec x \) has vertical asymptotes at \( x = \frac{\pi}{2} \) and \( x = \frac{3\pi}{2} \). It has minima (low points) at \((0, 1)\) and \((2\pi, 1)\). It has a maximum (high point) at \((\pi, -1)\).
III. How to Graph \( y = a \csc (bx - c) + d \) and \( y = a \sec (bx - c) + d \)

1. Write the guide function. For \( y = a \csc (bx - c) + d \) the guide function is \( y = a \sin (bx - c) + d \) and for \( y = a \sec (bx - c) + d \) the guide function is \( y = a \cos (bx - c) + d \).

2. Identify \( a, b, c, \) and \( d \).

3a. Find the amplitude of the guide function: \( \text{amp} = |a| \).

Note: Cosecant and secant do not have amplitude.

b. Determine if there has been an x-axis reflection: if \( a \) is negative, there has been an x-axis reflection.

4a. Find the period: \( \text{period} = \frac{2\pi}{b} \).

b. Find the x-increment: \( \text{x-increment} = \frac{1}{4} \cdot \text{period} \).

5. Find the phase shift by setting the argument, \( bx - c \), equal to zero and solving for \( x \):

\[
P.S. = \frac{c}{b} \right \text{right (if we see } bx - c \text{ and } \frac{c}{b} \text{ is positive)} \text{ or} \\
P.S. = \frac{c}{b} \left \text{left (if we see } bx + c \text{ and } \frac{c}{b} \text{ is negative).}
\]

6. Find the vertical shift: \( |d| \up \text{ if } d \text{ is positive or } |d| \down \text{ if } d \text{ is negative.} \)

7. Identify the five key points of the guide function:

For sine: \( \left( \frac{c}{b}, d \right) \quad (x_1 + \text{x-inc}, d+a) \quad (x_2 + \text{x-inc}, d) \quad (x_3 + \text{x-inc}, d-a) \quad (x_4 + \text{x-inc}, d) \)

For cosine: \( \left( \frac{c}{b}, d+a \right) \quad (x_1 + \text{x-inc}, d) \quad (x_2 + \text{x-inc}, d-a) \quad (x_3 + \text{x-inc}, d) \quad (x_4 + \text{x-inc}, d+a) \)

8. Plot the five key points of the guide function and connect them with a smooth, continuous, dotted curve, then extend the graph to two full periods.

9. Draw the given function using these guidelines:

Anywhere the guide function crosses its horizontal axis (i.e., when \( y = d \)), the given function has a vertical asymptote. Anywhere the guide function has a minimum, the given function has a maximum and we draw a parabola shaped curve opening down. Anywhere the guide function has a maximum, the given function has a minimum, and we draw a parabola shaped curve opening up.
Example 1:  \( y = 2 \csc (x - 3\pi) - 1 \)

1. Guide function:  \( y = 2 \sin (x - 3\pi) - 1 \)
2. \( a = 2 \quad b = 1 \quad c = 3\pi = \frac{6\pi}{2} \quad d = -1 \)
3. \( \text{amp of sine} = |2| = 2 \quad \text{There is no x-axis reflection.} \)
4. \( \text{period} = \frac{2\pi}{1} = 2\pi \quad \text{x-increment} = \frac{1}{4} \cdot 2\pi = \frac{2\pi}{4} = \frac{\pi}{2} \)
5. \( \text{phase shift} = 3\pi \) or \( \frac{6\pi}{2} \) right \( x - 3\pi = 0 \rightarrow x = 3\pi \)
6. \( \text{vertical shift} = 1 \downarrow \)
7. 5 key points of the sine:

\[
\begin{align*}
\left( \frac{\pi}{2}, -1 \right) & \quad \left( \frac{7\pi}{2}, 1 \right) & \quad \left( \frac{8\pi}{2}, -1 \right) & \quad \left( \frac{9\pi}{2}, -3 \right) & \quad \left( \frac{10\pi}{2}, -1 \right)
\end{align*}
\]

Example 2:  \( y = -\frac{1}{4} \sec \left( x + \frac{\pi}{2} \right) + 3 \)

1. Guide function:  \( y = -\frac{1}{4} \cos \left( x + \frac{\pi}{2} \right) + 3 \)
2. \( a = -\frac{1}{4} \quad b = 1 \quad c = -\frac{\pi}{2} \quad d = 3 = \frac{12}{4} \)
3. \( \text{amp of cosine} = \left| -\frac{1}{4} \right| = \frac{1}{4} \quad \text{There is an x-axis reflection.} \)
4. \( \text{period} = \frac{2\pi}{1} = 2\pi \quad \text{x-increment} = \frac{1}{4} \cdot 2\pi = \frac{2\pi}{4} = \frac{\pi}{2} \)
5. \( \text{phase shift} = \frac{\pi}{2} \) left \( x + \frac{\pi}{2} = 0 \rightarrow x = -\frac{\pi}{2} \)
6. \( \text{vertical shift} = 3 \) or \( \frac{12}{4} \) up
7. 5 key points of the cosine:

\[
\begin{align*}
\left( -\frac{\pi}{2}, \frac{11}{4} \right) & \quad \left( 0, \frac{12}{4} \right) & \quad \left( \frac{\pi}{2}, \frac{13}{4} \right) & \quad \left( \frac{2\pi}{2}, \frac{12}{4} \right) & \quad \left( \frac{3\pi}{2}, \frac{11}{4} \right)
\end{align*}
\]
IV. The Basic Tangent Function

The graph of \( y = \tan x \) has a period of \( \pi \) and an \( x \)-increment of \( \frac{\pi}{4} \). It is an odd function with origin symmetry. Because \( \tan x = \frac{\sin x}{\cos x} \), the tangent function is undefined whenever \( \cos x = 0 \).

Thus the domain of \( \tan x \) is all real numbers except odd multiples of \( \frac{\pi}{2} \). At \( x = \frac{\pi}{2} + n \cdot \pi \), where \( n \) is an integer, the graph of \( y = \tan x \) has a vertical asymptote. The range of \( \tan x \) is \( (-\infty, \infty) \).

Notice that the graph of \( y = \tan x \) is increasing along its entire length. Its left “hand” points down and its right hand points up. The 5 key features of the graph of \( y = \tan x \) are:

left asymptote: \( x = -\frac{\pi}{2} \)  
3 key points: \( \left( -\frac{\pi}{4}, -1 \right) \)  \( (0, 0) \)  \( \left( \frac{\pi}{4}, 1 \right) \)  
right asymptote: \( x = \frac{\pi}{2} \)

V. The Basic Cotangent Function

The graph of \( y = \cot x \) has a period of \( \pi \) and an \( x \)-increment of \( \frac{\pi}{4} \). It is an odd function with origin symmetry. Because \( \cot x = \frac{\cos x}{\sin x} \), the cotangent function is undefined whenever \( \sin x = 0 \).

Thus the domain of \( \cot x \) is all real numbers except integer multiples of \( \pi \). At \( x = n \cdot \pi \), where \( n \) is an integer, the graph of \( y = \cot x \) has a vertical asymptote. The range of \( \cot x \) is \( (-\infty, \infty) \).

Notice that the graph of \( y = \cot x \) is decreasing along its entire length. Its left “hand” points up and its right hand points down. The 5 key features of the graph of \( y = \cot x \) are:
left asymptote: x = 0
3 key points: \( \left( \frac{\pi}{4}, 1 \right), \left( \frac{\pi}{2}, 0 \right), \left( \frac{3\pi}{4}, -1 \right) \) 
right asymptote: x = \( \pi \)

VI. How to Graph \( y = a \tan{(bx – c)} + d \) and \( y = a \cot{(bx – c)} + d \)

1. Identify a, b, c, and d.
2. Determine if there has been an x-axis reflection: if a is negative, there has been an x-axis reflection.
3a. Find the period: period = \( \frac{\pi}{b} \).
   b. Find the x-increment: x-increment = \( \frac{1}{4} \cdot \frac{\pi}{\text{period}} \).
4. Find the phase shift by setting the argument, \( bx – c \), equal to zero and solving for x:
   \[ \text{P.S.} = \frac{c}{b} \text{ right (if we see } bx - c \text{ and } \frac{c}{b} \text{ is positive) or} \]
   \[ \text{P.S.} = -\frac{c}{b} \text{ left (if we see } bx + c \text{ and } -\frac{c}{b} \text{ is negative).} \]
5. Find the vertical shift: |d| up if d is positive or |d| down if d is negative.
6. Identify the left asymptote, the three key points, and the right asymptote.

**cosecant:** To find the left asymptote, let x = phase shift, \( x = \frac{c}{b} \).

LA: \( x = \frac{c}{b} \) (x of LA + x-increment, d + a) \( (x_1 + \text{x-inc, d}) \) \( (x_2 + \text{x-inc, d} - a) \) RA: \( x = x_3 + \text{x-inc} \)

**Note:** The left asymptote of a tangent function is NOT x = phase shift.

**tangent:** To find the left asymptote, set the argument, \( bx – c \), equal to \(-\frac{\pi}{2}\) and solve for x.

LA: \( x = \frac{c}{b} - \frac{\pi}{2b} \) (x of LA + x-inc, d – a) \( (x_1 + \text{x-inc, d}) \) \( (x_2 + \text{x-inc, d} + a) \) RA: \( x = x_3 + \text{x-inc} \)

7. Plot the asymptotes and the three key points. Connect the points with a smooth, continuous curve. Extend the graph to two full periods.
Example 3: \( y = 2 \tan \left( \frac{1}{4}x - \pi \right) - 3 \)

1. \( a = 2 \quad b = \frac{1}{4} \quad c = \pi \quad d = -3 \)
2. There is no x-axis reflection.
3. \( \text{period} = \frac{\pi}{\frac{1}{4}} = 4\pi \quad \text{x-increment} = \frac{1}{4} \cdot 4\pi = \pi \)
4. \( \text{phase shift} = 4\pi \text{ right} \quad \frac{1}{4}x - \pi = 0 \rightarrow x - 4\pi = 0 \rightarrow x = 4\pi \)
5. \( \text{vertical shift} = 3 \text{ down} \)
6. LA: \( \frac{1}{4}x - \pi = -\frac{\pi}{2} \rightarrow x - 4\pi = -2\pi \rightarrow x = 2\pi \quad (3\pi, -5) \quad (4\pi, -3) \quad (5\pi, -1) \quad \text{RA: } x = 6\pi \)

Example 4: \( y = -3 \cot (2x + \pi) + 1 \)

1. \( a = -3 \quad b = 2 \quad c = -\pi \quad d = 1 \)
2. There is an x-axis reflection.
3. \( \text{period} = \frac{\pi}{2} \quad \text{x-increment} = \)
4. \( \text{phase shift} = \frac{\pi}{2} \text{ left} \quad 2x + \pi = 0 \rightarrow 2x = -\pi \rightarrow x = -\frac{\pi}{2} = -\frac{4\pi}{8} \)
5. \( \text{vertical shift} = 1 \text{ up} \)
6. LA: \( x = -\frac{\pi}{2} = -\frac{4\pi}{8} \quad \left( -\frac{3\pi}{8}, -2 \right) \quad \left( -\frac{2\pi}{8}, 1 \right) \quad \left( -\frac{\pi}{8}, 4 \right) \quad \text{RA: } x = 0 \)
Section 4.4 Harmonic Motion

I. Definition

Simple harmonic motion is a periodic, oscillating movement which can be described by a sinusoidal function. Examples of physical phenomena which can be described as simple harmonic motion include radio and television waves; light waves; sound waves; water waves; the swinging of a pendulum; the vibrations of a tuning fork; and the bobbing of a weight attached to a coiled spring.

II. The Movement of a Mass on a Spring

Consider a mass on a spring. The system is said to be in equilibrium when the mass is at rest. The point of rest is called the origin of the system. We consider the distance above equilibrium as positive and the distance below equilibrium as negative.

If the mass is lifted a distance \(a\) and released, it will oscillate up and down in periodic motion. If there is no friction, the motion repeats itself in a certain period of time. The distance \(a\) is called the displacement from the origin. The number of times the mass oscillates in 1 unit of time is called the frequency \(f\) of the motion, and the time it takes for the mass to complete one oscillation is called its period. The frequency and period are related by the formulas:

\[
\text{period} = \frac{1}{f} \quad \text{and} \quad f = \frac{1}{\text{period}}
\]

and \(\text{period} = \frac{1}{f}\). The maximum displacement from equilibrium is called the amplitude of the motion.

III. Equations Modeling Simple Harmonic Motion

If the displacement from the origin is at a maximum at time \(t = 0\), simple harmonic motion can be modeled by \(y = a \cos \omega t\).

If the displacement from the origin is zero at time \(t = 0\), simple harmonic motion can be modeled by \(y = a \sin \omega t\).

For either model, |\(a\)| = amplitude (maximum displacement); \(p = \text{period} = \frac{2\pi}{\omega}\);

\[
f = \text{frequency} = \frac{\omega}{2\pi}; \quad \omega = 2\pi f \quad \text{or} \quad \omega = \frac{2\pi}{p}; \quad t \text{ is the time, and } y \text{ is the displacement at time } t.
\]
Example 1  An object is attached to a coiled spring. It is pulled down a distance of 6 units from its equilibrium position, and then released. The time for one complete oscillation is 4 sec. (#10)

a. Give an equation that models the position of the object at time t.

\[ a = -6 \quad p = 4 \quad \omega = \frac{2\pi}{4} = \frac{\pi}{2} \quad y = -6\cos\left(\frac{\pi}{2} \cdot t\right) \]

b. Determine the position at \( t = 1.25 \) sec.

\[ y = -6\cos\left(\frac{\pi}{2} \cdot (1.25)\right) = 2.30 \text{ units} \]

c. Find the frequency.

\[ f = \frac{1}{p} = \frac{1}{4} \]

The formula for the up and down motion of a weight on a spring is given by \( y = a\sin\left(\sqrt{\frac{k}{m}} \cdot t\right) \) where \( k \) is the spring constant and \( m \) is the mass of the weight.

Example 2  A spring with a spring constant of \( k = 3 \) has a 27 unit mass suspended from it.

a. If the spring is stretched 1.5 ft and released, what are the amplitude, period and frequency of the resulting oscillatory motion?

\[ a = 1.5; \quad b = \sqrt{\frac{3}{27}} = \frac{1}{3}; \quad \text{amp} = 1.5; \quad \text{period} = \frac{2\pi}{\sqrt{3}} = 6\pi; \quad \text{frequency} = \frac{1}{6\pi} \]

b. What is the equation of the motion?

\[ y = \frac{3}{2}\sin\left(\frac{1}{3} \cdot t\right) \]

Example 3  The height of a weight attached to a spring is \( y = -4\cos(8\pi t) \) inches after \( t \) seconds. (Ch 4 Test #9)

a. Find the maximum height that the weight rises above the equilibrium position of \( y = 0 \).

max height = amplitude = 4 inches

b. When does the weight first reach it maximum height, if \( t \geq 0 \)?

\[ 4 = -4\cos(8\pi t) \rightarrow -1 = \cos(8\pi t) \rightarrow \cos \theta = -1 \text{ when } \theta = \pi \rightarrow 8\pi t = \pi \rightarrow t = \frac{\pi}{8\pi} = \frac{1}{8} \text{ second} \]

c. What are the frequency and period?

\[ f = \frac{b}{2\pi} = \frac{8\pi}{2\pi} = 4 \text{ cycles per second}; \quad \text{period} = \frac{1}{4} \text{ second} \]