Finding Square Roots.

Recall:

When we raise a number to the *second power*, $c^2$, we say it is *squared*.

- The *square* of a number is the number times itself.  
  In symbols, $a^2 = a \cdot a$.

For example,

- The square of 5 is 25 because $5^2 = 5 \cdot 5 = 25$.
- The square of -5 is $(-5)^2 = (-5)(-5) = 25$.

When we want to *find what number was squared*, we are finding a square root (the inverse of squaring).

- The inverse of squaring is finding a square root.

For example,

- One square root of 25 is 5 because $5^2 = 25$.
- The other square root of 25 is -5 because $(-5)^2 = 25$, too.

- Squaring and finding a square root are inverse operations.

- Every positive number will have two real-number square roots, (one positive and one negative).
  The number 0 (zero) has just one square root, 0 itself.

Example 1  
Find the square roots of 121.  
The square roots of 121 are 11 and -11 because $11^2 = 121$ and $(-11)^2 = 121$.

Example 2  
Find the square root of 0.  
The only square root of 0 is 0 (since 0 is not positive or negative, so those choices don’t exist here).

Now Do Practice Exercises 1 - 5.
1. Squaring and finding a square root are operations.
Find all square roots for each of the following.
2. 9 3. 0 4. 100 5. 144

- Radical Notation: $\sqrt{a}$

The pieces of a radical expression for square roots: $\sqrt{a}$, are

the radical sign or radical: $\sqrt{}$, and the radicand, $a$. (NOTE: The radicand is the entire expression under the radical, even if it’s composed of several terms or factors).
Radical Notation:

If \( a > 0 \) then \( \sqrt{a} = b, \quad b > 0 \) and if \( b^2 = a \).

In words, if \( a \) is a positive number, \( \sqrt{a} \) is the **positive square root** or **principal square root** of \( a \) and \( \sqrt{a} \) is equal to the **positive number** \( b \) whose square is \( a \). \( \sqrt{a} \) is read “the square root of \( a \”).

When the radical \( \sqrt{\phantom{0}} \) sign is used, to ask for the **negative square root**, a negative sign must be written in front of the radical:

\[-\sqrt{a} = -b, \quad b > 0 \quad \text{and if} \quad b^2 = a.\]

Also note:

\[\sqrt{0} = 0.\]

**Example 3**

Find each square root.

a. \( \sqrt{49} \)

Solution: \( \sqrt{49} = 7 \) because \( 7^2 = 49 \).

b. \( \sqrt{1} \)

Solution: \( \sqrt{1} = 1 \) because \( 1^2 = 1 \).

**Example 4**

Find:

\[\frac{4}{\sqrt{25}}\]

Solution: \( \frac{4}{\sqrt{25}} = \frac{2}{5} \) because \( \left( \frac{2}{5} \right)^2 = \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25} \)

**Example 6**

a) Find two whole values to approximate \( \sqrt{23} \) between.

b) Use a calculator to approximate the \( \sqrt{23} \) to the nearest thousandth.

---

**Approximating Square Roots.**

So far, we’ve looked at square roots of perfect squares. Numbers like \( \frac{1}{4}, 36, \frac{7}{25}, \) and 1 are called **perfect squares** because they’re the square of a whole number (or a fraction) and their square root is **rational** (see part C below for the formal definition of rational). A square root such as \( \sqrt{5} \) cannot be written as a whole number since 5 isn’t a perfect square. Such numbers are called **irrational** (see part C below for the formal definition of irrational).

Although \( \sqrt{5} \) can't be written as a whole number, it can be approximated between two nearest wholes by finding the perfect squares surrounding the radicand or it can be approximated to about 12 decimal places **using a calculator**. Since 5 is between 4 and 9, \( \sqrt{5} \) is between \( \sqrt{4} \) and \( \sqrt{9} \), i.e., \( \sqrt{5} \) is between 2 and 3.

Using a calculator, \( \sqrt{5} \approx 2.236067977 \). **Caution:** these are approximations and are not the exact value of \( \sqrt{5} \). The exact value for \( \sqrt{5} \) cannot be written as a whole number or a decimal, as it doesn't repeat or terminate, and is most simply written **exactly** as its radical form: \( \sqrt{5} \).
Solution: a) Since 23 is between 16 and 25, \( \sqrt{23} \) is between \( \sqrt{16} \) and \( \sqrt{25} \), or \( \sqrt{23} \) is between 4 and 5 (but closer to 5).

On a scientific calculator, the square root key is usually a 2\(^{nd}\) function above the \( x^2 \) (square) key.

For a scientific calculator (based on a TI-30), press: ON, 4, 3, 2\(^{nd}\), \( x^2 \) (and maybe =).

For a graphing calculator (based on a TI-82/83), you press: ON, 2\(^{nd}\), \( x^2 \), 4, 3, Enter.

You should see 4.795831523 appear.

NOTE: Most calculators are accurate to at least 3 more places than they display. The three more places for this problem are shown here: \( \sqrt{23} \approx 4.795831523127 \).

Since the nearest thousandth means 3 places after the decimal (or 3 decimal places), we look at the place to the right of that, and notice its an 8, which is greater than 5. So round the thousandths place up and drop values right of that.

**To the nearest thousandth**, \( \sqrt{23} \approx 4.796 \)

Caution: \( \sqrt{23} = 4.796 \) is not correct, since it is an approximation. Whenever you round from your calculator, you should use the approximately equals symbol, \( \approx \). So \( \sqrt{23} \approx 4.796 \) is the correct notation. The only exact way to write \( \sqrt{23} \) is \( \sqrt{23} \), (or by using rational exponents – these will be discussed in future courses).

Now Do Practice Exercises 14 – 17.
14. Find two whole values to approximate \( \sqrt{29} \) between.
15. Approximate \( \sqrt{29} \) to the nearest thousandth.
Determine if each of the following is true or false. State a reason in either case.
16. The number \( \sqrt{5} = 2.23 \).
17. The number \( \sqrt{64} = 8 \).

C More on Rational vs. Irrational Numbers:

A **rational number** \( q \) is any number that can be written as a **ratio of integers** (or quotient or fraction of integers), where the denominator cannot be 0.

In **set-builder notation**: \( \left\{ q \mid q = \frac{a}{b}, \text{ where both } a \text{ and } b \text{ are integer and } b \neq 0 \right\} \)

(read as “the set of numbers \( q \) such that \( q \) is equal to \( a \) divided by \( b \), where both \( a \) and \( b \) are integer and \( b \) is not equal to zero”).

The **decimal** form of a rational number always either **terminates** (i.e., \( \frac{1}{3} = 0.3333... \), repeating the 3 forever).

Note: any **rational number** can always be written as a **fraction composed of integers**.

You may recall, if it is a terminating decimal, you put all the digits after the decimal over the digit 1 followed by that many zeros.

Example: For 0.75, put 75 over 1 followed by two 0’s or 100 (and reduce). \( 0.75 = \frac{75}{100} = \frac{3}{4} \).

If it’s a repeating decimal, put exactly one full repeat of it’s digits over that same number of 9’s.

Example: For 0.090909... = 0.0909, put one full repeat (09) over two 9’s (and reduce).

\[ 0.0909 = \frac{09}{99} = \frac{9}{99} = \frac{1}{11} \] once it’s reduced.
More examples of rational numbers written in fraction and decimal form:
\[ \frac{1}{3} = 0.333\ldots, \quad 2 = \frac{2}{1}, \quad \frac{3}{4} = 0.75, \quad \sqrt{9} = 3 = \frac{3}{1}, \quad 0 = \frac{0}{1} \]

Caution: some rational number decimal repeats cannot be seen as easily and cannot be seen on a calculator at all. For example, the number \( \frac{1}{17} = 0.0588235294117605882352941176 \) has 16 digits in a single repeat, but because \( \frac{1}{17} \) is a fraction of integers, it obeys the definition and is a rational number.

Reminder, since any natural number, whole number, or integer can be put over 1, all of these are also in the set of rational numbers.

An irrational number is any number that is not rational. It cannot be written as a ratio of integers and its decimal representation neither terminates nor repeats.

Some examples of irrational numbers written in exact and approximately equal (=) decimal form:
\[ \sqrt{2} = 1.414213562\ldots, \quad \pi = 3.141592654\ldots, \quad \sqrt{10} = 3.16227766\ldots, \quad 0.101001000100001\ldots \] (This last one cannot be written in an exact form.)

NOTE: To get an accurate decimal form of any irrational number, use a scientific or graphing calculator, as they are not easily found by hand.

Caution: numbers like \( \sqrt{9} \), which is exactly equal to \( 3 = \frac{3}{1} \), and \( \frac{1}{17} \), whose decimal repeat is long, are both still rational numbers, since they obey that definition (they can be written as a ratio of integers).

NOTE: Most square roots are irrational. Only square roots of perfect squares are rational.

The easiest way to tell if a number is irrational is to check that it is not rational. Ask your self these questions: Can it be written as some ratio (fraction) of integers? Or, if it’s hard to tell that, check (on a calculator), does the decimal representation clearly terminate or repeat? (Again, these may not work if the repeat is a long one.)

Now Do Practice Exercises 18 – 20.

Determine if the following numbers are rational or irrational. Show your reasoning in either case.
18. \( \frac{81}{7} \)
19. \( \sqrt{43} \)
20. \( \sqrt{9} \)

Exercises for the Radicals Packet, Part 1:

\[ \text{a} \] Find all square roots.
1. \( 144 \)

Find each square root.
2. \( \sqrt{121} \)
3. \( -\sqrt{9} \)

\[ \text{b} \] Find two whole values to approximate each square root between.
4. \( \sqrt{15} \)
5. \( \sqrt{45} \)

Use a calculator to approximate each square root. Round the square root to the nearest thousandth.
6. \( \sqrt{7} \)
7. \( \sqrt{1100} \)

\[ \text{c} \] Determine if each of the following numbers is rational or irrational.
8. \( \frac{4}{\sqrt{9}} \)
9. \( \sqrt{7} \)
10. \( 0.124124124124\ldots \)
Square Roots, Reviewed and Expanded.

Recall:
When we raise a number to the second power, \( c^2 \), we say it is squared. When we want to find what number was squared, we are finding a square root.

\( \sqrt{-4} \) is not a real number - because there’s no real number whose square is \(-4\); i.e., there is no real number \( x \) such that \( x^2 = -4 \) (since the square of any real number multiplies an even number of negatives, and is therefore always going to be positive (or zero)).

Square roots of negatives are imaginary numbers and will be discussed in later courses. Caution: they are not undefined, as they do exist and are used in some very real applications, e.g., the field of electronics.)

Radical Notation expanded:
The (principal) square root of a positive number \( a \) is the positive number \( b \) whose square is \( a \). In symbols, if both \( a > 0 \) and \( b > 0 \),
\[ \sqrt{a} = b \text{ if } b^2 = a, \]
\[ -\sqrt{a} = -b \text{ if } b^2 = a, \]
\[ \sqrt{0} = 0, \]
and \( \sqrt{-a} \) is not a real number.

Example 1 Evaluate each of the following.

\begin{align*}
\text{a. } & \sqrt{49} \quad \text{b. } \sqrt[4]{\frac{4}{25}} \quad \text{c. } \sqrt{-81} \\
\text{Solution: } & \quad \text{a. } \sqrt{49} = 7 \quad \text{because } 7^2 = 49. \\
\text{Solution: } & \quad \text{b. } \sqrt[4]{\frac{4}{25}} = \frac{2}{5} \quad \text{because } \left(\frac{2}{5}\right)^2 = \frac{2}{5} \cdot \frac{2}{5} = \sqrt[4]{\frac{4}{25}}. \\
& \quad \text{c. } \sqrt{-81} \text{ is not real } \quad \text{because } \text{ there is no real number whose square is } -81. \\
\end{align*}

Now Do Practice Exercises 1 - 4.

Evaluate.
\begin{align*}
1. & \sqrt{-100} \quad 2. \sqrt{0} \quad 3. \sqrt[4]{\frac{1}{4}} \quad 4. \sqrt{.01} \quad \left( \text{Hint, recall } .01 \text{ is } \frac{1}{100} \right)
\end{align*}
**Nth roots.**

As noted earlier, finding the square root is the inverse of squaring a number. Here we'll extend that idea to work with other roots of numbers.

For example, the **cube root** of a number is the number we must cube (raise to the third power) to get that number. (The cube root of 8 is 2 since \(2^3 = 8\), and we write \(\sqrt[3]{8} = 2\).)

The **fourth root** of a number is the number we must raise to the fourth power to get that number, etc.

The **\(n^{th}\) root** of \(a\), written \(\sqrt[n]{a}\), is the number \(x\) where \(x^n = a\).

As before, each part of a radical expression has a special name. The parts of a radical expression for any root, are the **index**, \(n\), (the “root” number) \(\sqrt[n]{a}\), the **radical sign** or **radical**, \(\sqrt{\phantom{0}}\), and the **radicand**, \(a\). (Recall, the **radicand** is the entire expression under the **radical**, even if it’s composed of several terms or factors).

For example,

The **cube root** of 64 is written \(\sqrt[3]{64}\) (Index is 3) and it represents the number that would be cubed (or raised to the third power) to get 64.

So, \(\sqrt[3]{64} = 4\) because \(4^3 = 64\).

The **fourth root** of 81 is written \(\sqrt[4]{81}\) (Index is 4) It represents the number we would raise to the fourth power to get 81.

So, \(\sqrt[4]{81} = 3\) because \(3^4 = 3\cdot3\cdot3\cdot3 = 81\).

**NOTE:** When the index is 2, it is a **square root** and it’s not written: \(\sqrt[2]{a} = \sqrt{a}\)

**Odd roots** of negative numbers, (where the index is odd, will be both real and negative.

(This is true since an odd power multiplies an odd number of negatives, and is therefore always going to be negative. So a negative real root will exist in these cases.)

For example, \(\sqrt[3]{-64} = -4\) because \((-4)^3 = -64\).

**Even roots** of roots of negative numbers, (where the index is even), are still not real numbers.

For example, \(\sqrt[4]{-16}\) is not real because there is no real number whose fourth power is \(-16\); i.e., there is no real number that can be raised to the fourth power and will result in a negative number.

This is because an even power of any real number multiplies an even number of negatives, and is therefore always going to be positive or zero.
The following table shows the most commonly used roots. You fill in the missing values.

<table>
<thead>
<tr>
<th>Square Roots</th>
<th>Cube Roots</th>
<th>Fourth Roots</th>
<th>Fifth Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{1} = 1$</td>
<td>$\sqrt[3]{1} = 1$</td>
<td>$\sqrt[4]{1} = 1$</td>
<td>$\sqrt[5]{1} = 1$</td>
</tr>
<tr>
<td>$\sqrt{4} = 2$</td>
<td>$\sqrt[3]{64} = 4$</td>
<td>$\sqrt[4]{16} = 2$</td>
<td>$\sqrt[5]{32} = 2$</td>
</tr>
<tr>
<td>$\sqrt{25} = 5$</td>
<td>$\sqrt[3]{125} = 5$</td>
<td>$\sqrt[4]{81} = 3$</td>
<td>$\sqrt[5]{243} = 3$</td>
</tr>
<tr>
<td>$\sqrt{36} = 6$</td>
<td>$\sqrt[3]{216} = 6$</td>
<td>$\sqrt[4]{64} = 4$</td>
<td>$\sqrt[5]{1024} = 4$</td>
</tr>
</tbody>
</table>

Example 2  Evaluate each of the following.

a. $\sqrt[4]{256} = 4$ because $4^4 = 256$

b. $\sqrt[5]{1024} = 4$ because $4^5 = 1024$

c. $\sqrt[3]{-125} = -5$ because $(-5)^3 = -125$

d. $\sqrt[4]{-256}$ is not real because there is no real number whose fourth power is $-256$.

Now Do Practice Exercises 5 – 8.
Evaluate if possible.
5. $\sqrt[3]{27}$
6. $\sqrt[4]{16}$
7. $\sqrt[4]{-81}$
8. $\sqrt[3]{-1}$

To conclude this portion, we develop a general result needed in later courses. Let’s start by looking at two examples.

$\sqrt{2^2} = \sqrt{4} = 2$, and $\sqrt{(-2)^2} = \sqrt{4} = 2$ since $(-2)^2 = 4$

Consider the value of $\sqrt{x^2}$ where $x$ is positive or negative.

In $\sqrt{2^2}$ where $x = 2$, $\sqrt{2^2} = 2$.

In $\sqrt{(-2)^2}$ where $x = -2$, $\sqrt{(-2)^2} \neq -2$.

Here $\sqrt{(-2)^2} = -(-2) = 2$ (the opposite of $-2$).

Comparing the results above, we see that

$\sqrt{x^2}$ is $x$ if $x$ is positive or zero, and $\sqrt{x^2}$ is $-x$ (the opposite of $x$) if $x$ itself is negative,

(i.e., raising to an even power, and then taking an even root forces all things positive or zero).

From your earlier work with absolute values you may remember that

$|x| = x$ if $x$ is positive or zero, and $|x| = -x$ (the opposite of $x$) if $x$ itself is negative.

(i.e., absolute value forces all things positive or zero).

Since this same sign pattern works for any even power and root, we can summarize the discussion by writing

$\sqrt[n]{x^n} = |x|$ for any even index, $n$, and any real number, $x$. 

Example 3 Evaluate each of the following.

a. \( \sqrt{5^2} \)  
   Solution: \( \sqrt{5^2} = 5 \) because \( \sqrt{25} = 5 \) or \( \sqrt{5^2} = |5| = 5 \)

b. \( \sqrt{(-4)^2} \)  
   Solution: \( \sqrt{(-4)^2} = 4 \) because \( \sqrt{16} = 4 \) or \( \sqrt{(-4)^2} = |-4| = 4 \)

Now Do Practice Exercises 9 – 10.

Evaluate.

9. \( \sqrt{7^2} \)  
10. \( \sqrt{(-7)^2} \)

NOTE: Roots with odd indices do not require the absolute value, since an odd power (and so an odd number of negative signs multiplied) is involved.

For example:
\[
\sqrt[3]{5^3} = \sqrt[3]{125} = 5, \quad \text{because} \quad 5^3 = 125
\]
and \( \sqrt[3]{(-5)^3} = \sqrt[3]{-125} = -5 \) because \( (-5)^3 = -125 \)

so

\[ \sqrt[n]{x^n} = x \quad \text{where the index, n, is odd.} \]

Combining the two concepts in one statement, we can say

\[ \sqrt[n]{x^n} = \begin{cases} 
  |x| & \text{where the index n is even} \\
  x & \text{where the index n is odd}
\end{cases} \]

Example 4 Evaluate each of the following.

a. \( \sqrt{(-4)^2} \)  
   Solution: \( \sqrt{(-4)^2} = |-4| = 4 \) because \( \sqrt[n]{x^n} = |x| \) when the index, n, is even (it’s the unwritten 2 ro a square root here)

b. \( \sqrt[3]{5^3} \)  
   Solution: \( \sqrt[3]{5^3} = 5 \) because \( \sqrt[n]{x^n} = x \) when the index, n, is odd (n = 3 here)

c. \( \sqrt[3]{(-5)^3} \)  
   Solution: \( \sqrt[3]{(-5)^3} = -5 \) because \( \sqrt[n]{x^n} = x \) when the index, n, is odd (n = 3 here)

Solution: d. \( \sqrt[4]{(-2)^4} = |-2| = 2 \) because \( \sqrt[n]{x^n} = |x| \) when the index, n, is even (n = 4 here)

Now Do Practice Exercises 11 – 13.

Evaluate.

11. \( \sqrt[3]{7^3} \)  
12. \( \sqrt[4]{(-2)^4} \)  
13. \( \sqrt[5]{(-1)^5} \)

C Simplifying Radical Expressions.

For most applications, we need to put answers with radical expressions in "simplest form". The word simplest here just means the following three conditions have been met (the actual result may not look any "simpler" than when you started).

A radical expression is in simplest form when

1. There are no perfect-power factors (greater than or equal to the index) inside a radical.
2. No fraction appears inside a radical.
3. No radical appears in the denominator of a fraction.
As we will be sticking to square roots in this course, this alters the first condition to read:

1. **There are no perfect-square factors under a square root.**

For example:
- \(\sqrt{9}\) is *not* simplest form, since 9 is a perfect-square factor, and
- \(\sqrt{8}\) is *not* simplest form, since \(\sqrt{8} = \sqrt{4 \cdot 2}\), and 4 is a perfect-square factor.

But \(\sqrt{7}\) is simplest form, since 7 has *no* perfect-square factor (other than 1).

We need to look at two properties to simplify radical expressions. The first of these will be used to simplify expressions with perfect square factors under square root. (These properties are directly related to the “product law” and “quotient law” for exponents you studied previously, as you will see in the next course.)

First, look at the following example.

\[
\sqrt{25 \cdot 4} = \sqrt{100} = 10 \\
\sqrt{25} \cdot \sqrt{4} = 5 \cdot 2 = 10
\]

This shows that \(\sqrt{25 \cdot 4} = \sqrt{25} \cdot \sqrt{4}\). Here is the general law for this fact.

### Product Property of Radicals

For any non-negative real numbers \(a, b\), and positive real number \(n\),

\[\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}\]

In words, the \(n\)th root of a product is the product of the \(n\)th roots (and visa versa).

Caution: \(\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}\) By letting \(a = 25\) and \(b = 4\), we can see why.

This makes the left side: \(\sqrt{25 + 4} = \sqrt{29}\),

whereas the right side becomes: \(\sqrt{25} + \sqrt{4} = 5 + 2 = 7\),

and clearly \(\sqrt{29} \neq 7\).

The first few perfect squares are listed here as a reminder to help you recognize them as factors:

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n^2)</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>36</td>
<td>49</td>
<td>64</td>
<td>81</td>
<td>100</td>
<td>121</td>
<td>144</td>
<td>169</td>
<td>196</td>
<td>225</td>
</tr>
</tbody>
</table>

**NOTE:** \(a\sqrt{b} = a \cdot \sqrt{b}\)  and  \(a \cdot \sqrt{b} = a\sqrt{b}\) (The operation is multiply between a number or variable and a radical.)

**Example 4** Simplify each of the following.

a. \(\sqrt{9}\)  
   Solution: \(\sqrt{9} = 3\) since 9 is a *perfect square factor* itself.

b. \(\sqrt{8}\)  
   Solution: \(\sqrt{8} = \sqrt{4 \cdot 2}\)  
   \(= \sqrt{4} \cdot \sqrt{2}\) since \(\sqrt{4} = 2\)  
   \(= 2 \cdot \sqrt{2}\) by the Product Property of Radicals, (note: 4 is a *perfect square factor*)  
   \(= 2\sqrt{2}\)

c. \(\sqrt{7}\) is already simplest form  
   since 7 has *no* perfect square factors (other than 1)
Solution:  
\[ \sqrt{108} \]
\[ = \sqrt{9 \cdot 12} \]
\[ = \sqrt{9} \cdot \sqrt{12} \]
\[ = \sqrt{9} \cdot \sqrt{4 \cdot 3} \]
\[ = \sqrt{9} \cdot \sqrt{3} \]
\[ = 3 \cdot \sqrt{3} \]
\[ = 6\sqrt{3} \]

since 108 is divisible by 9 and then 12 is further divisible by 4. 
by the Product Property of Radicals, 
(here, 9 and 4 are both perfect square factors) 
since \( \sqrt{9} = 3 \) and \( \sqrt{4} = 2 \).

NOTE: If you didn’t notice that 9 divides 108, you can always break the radicand down to it’s prime factors to simplify the radical. If you use this method, you want to remove factor pairs from under the radical (since they produce perfect square factors).

Alternate Solution:  
\[ \sqrt{108} \]
\[ = \sqrt{2 \cdot 2 \cdot 3 \cdot 3 \cdot 3} \]
\[ = \sqrt{2 \cdot 2 \cdot \sqrt{3} \cdot 3 \cdot 3} \]
\[ = \sqrt{2^2 \cdot 3^2 \cdot 3} \]
\[ = 2 \cdot 3 \cdot \sqrt{3} \]
\[ = 6\sqrt{3} \]

since the prime factorization of 108 = \( 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \)
by the Product Property of Radicals
since \( 2 \cdot 2 = 2^2 \) and \( 3 \cdot 3 = 3^2 \)
since \( \sqrt{x^n} = |x| \) for even \( n \).

Now Do Practice Exercises 14 – 17.

Simplify.

14. \( \sqrt{50} \)  
15. \( \sqrt{45} \)  
16. \( \sqrt{28} \)  
17. \( \sqrt{162} \)

This process also works for variable expressions.

For this portion, as it will always be given that variables represent non-negative real numbers, the general nth root of \( x^n \) becomes more simple:

For any non-negative real number \( x \) and any integer \( n > 1 \), \[ \sqrt[n]{x^n} = x. \]

Example 5 Simplify each of the following. Assume that all variables represent non-negative real numbers.

a. \( \sqrt{4x^6} \)  
b. \( \sqrt{125b^3} \)  
c. \( \sqrt{108a^5} \)

Solution:  
a. \( \sqrt{4x^6} = \sqrt{4 \cdot x^2 \cdot x^2 \cdot x^2} \)
\[ = \sqrt{4} \cdot \sqrt{x^2} \cdot \sqrt{x^2} \cdot \sqrt{x^2} \]
\[ = 2 \cdot x \cdot x \cdot x \]
\[ = 2x^3 \]

since \( x^{m+n} = x^m \cdot x^n \) (product law of exponents)
by the Product Property of Radicals
since \( x \) is assumed to be non-negative, \( \sqrt[n]{x^n} = x \) for any \( n \).

b. \( \sqrt{125b^3} = \sqrt{25 \cdot 5 \cdot b^2 \cdot b} \)
\[ = \sqrt{25} \cdot \sqrt{5} \cdot \sqrt{b^2} \cdot \sqrt{b} \]
\[ = 5 \cdot \sqrt{5} \cdot b \cdot \sqrt{b} \]
\[ = 5b\sqrt{5b} \]
\[ = 5b\sqrt{5b} \]

since \( b^{m+n} = b^m \cdot b^n \) (product law of exponents)
by the Product Property of Radicals
since \( \sqrt{25} = 5 \) and \( b \) is non-negative, \( \sqrt[n]{b^n} = b \) for any \( n \).
by commuting.
by the Product Property of Radicals.
Solution:  
\[ \sqrt{108a^5} \]
Write the radicand as a product of squares times non-squares.

\[ = \sqrt{9 \cdot 4 \cdot 3 \cdot a^2 \cdot a^2 \cdot a} \]
since 108 = 9 \cdot 4 \cdot 3 by earlier work,

\[ = \sqrt{9} \cdot \sqrt{4} \cdot \sqrt{3} \cdot \sqrt{a^2} \cdot \sqrt{a^2} \cdot \sqrt{a} \]
and \( a^{m+n} = a^m \cdot a^n \) (product law of exponents used repeatedly)

\[ = 3 \cdot 2 \cdot \sqrt{3} \cdot a \cdot a \cdot \sqrt{a} \]
since \( \sqrt{9} = 3, \sqrt{4} = 2, \)

and \( \sqrt{a^n} = a \) when \( a \) is non-negative

\[ \]
(simplify the squared parts)

\[ = 3 \cdot 2 \cdot a \cdot a \cdot \sqrt{3} \cdot \sqrt{a} \]
by commuting (to put any radicals at the right end)

\[ = 6a^2 \sqrt{3a} \]
write the repeated factors as an exponent,

and put the 3 and \( a \) under one radical

(uses the Product Property of Radicals in reverse)

Now Do Practice Exercises 18 – 20.
Simplify. Assume that all variables represent non-negative real numbers.
18. \( \sqrt{9x^4} \) 19. \( \sqrt{8c^3} \) 20. \( \sqrt{75m^5} \)

So far we’ve only dealt with the first condition for simplest form (no square factors inside a square root). Before working on problems involving the second and third conditions, we’ll need to look at another property.

Look at the following two expressions:

\[ \frac{\sqrt{16}}{\sqrt{100}} = \frac{\sqrt{4}}{\sqrt{25}} = \frac{2}{5} \] (by earlier work)

and \( \frac{\sqrt{16}}{\sqrt{100}} = \frac{4}{10} = \frac{2}{5} \).

Thus, \( \frac{\sqrt{16}}{\sqrt{100}} = \frac{\sqrt{16}}{\sqrt{100}} \) which gives us the next general rule.

\[ \textbf{Quotient Property of Radicals} \]
For any non-negative real number \( a \), and positive real numbers \( b \) and \( n \),

\[ \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \]

In words, the \( n \)th root of a quotient is the quotient of the \( n \)th roots (and visa versa).

This property will be used to simplify radicals involving fractions to meet the last two conditions.

A radical expression is in simplest form when

2. **No fraction may remain inside a radical.**

3. **No radical may remain in the denominator of a fraction.**

Example 6 Simplify. Assume that all variables represent non-negative real numbers.

(Remember, that just means that \( \sqrt{x^2} = x \) here, instead of \( |x| \).)

\[ \]
a. \( \frac{\sqrt{16}}{\sqrt{4}} \)  

b. \( \frac{\sqrt{2}}{\sqrt{49}} \)  

c. \( \frac{\sqrt{12x^2}}{\sqrt{25}} \)
Solution: a. \[ \sqrt{\frac{16}{4}} = \frac{\sqrt{16}}{\sqrt{4}} \] by the Quotient Property of Radicals

We're not finished, since now we have a radical in the denominator.

\[ = \frac{4}{2} \] since \( \sqrt{16} = 4 \) and \( \sqrt{4} = 2 \)

We're still not done, since this reduces.

\[ = 2 \] since \( 4 \div 2 = 2 \)

NOTE, since 16 is divisible by 4, we could have performed that step first, (using the order of operations rules).

Alternate Solution: \[ \frac{\sqrt{16}}{\sqrt{4}} = \frac{\sqrt{16}}{\sqrt{4}} \] since \( 16 \div 4 = 4 \) and both are inside the radical.

\[ = 2 \] since \( \sqrt{4} = 2 \)

Solution: b. \[ \sqrt{\frac{2}{49}} = \frac{\sqrt{2}}{\sqrt{49}} \] by the Quotient Property of Radicals

We're not finished, since now there's a radical in the denominator.

\[ = \frac{\sqrt{2}}{7} \] since \( \sqrt{49} = 7 \)

Solution: c. \[ \sqrt{\frac{12x^2}{25}} = \frac{\sqrt{12x^2}}{\sqrt{25}} \] by the Quotient Property of Radicals

\[ = \frac{\sqrt{4 \cdot 3 \cdot x^2}}{\sqrt{25}} \] since \( 12 = 4 \cdot 3 \)

\[ = \frac{\sqrt{4} \cdot \sqrt{3} \cdot \sqrt{x^2}}{\sqrt{25}} \] by the Product Property of Radicals

\[ = \frac{2 \cdot \sqrt{3} \cdot x}{5} \] since \( \sqrt{4} = 2, \sqrt{25} = 5 \) and \( x \) is non-negative, \( \sqrt{x^n} = x \) for any \( n \).

\[ = \frac{2x\sqrt{3}}{5} \] by commuting

Now Do Practice Exercises 21 – 23.
Simplify. Assume that all variables represent non-negative real numbers.

21. \[ \sqrt{\frac{25}{16}} \] 22. \[ \sqrt{\frac{7}{9}} \] 23. \[ \sqrt{\frac{50x^2}{49}} \]

You may have noticed the above examples all had perfect squares in the denominator. If the denominator is not a perfect square, we will need to apply both the Quotient Property of Radicals and the Product Property of Radicals in a process called “rationalizing the denominator”. This involves multiplying by the denominator over itself (still as a radical) to force the denominator to be a perfect square factor under the radical. This technique rewrites the fraction as its equivalent with a rational number in the denominator (and thus satisfying condition 3 – leave no radical in the denominator.)

12
For Example:

To simplify the following expression, \( \frac{1}{\sqrt{2}} \), first use the Quotient Property of Radicals to rewrite the fraction under the radical as a quotient of radicals:

\[
\frac{1}{\sqrt{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}.
\]

This is *not* simplest form, since there is a radical in the denominator (see condition 3). To remove it, we need to multiply by a factor that will make the denominator a perfect square. Here, multiplying by \( \sqrt{2} \) would do the trick, since \( \sqrt{2} \cdot \sqrt{2} = \sqrt{4} = 2 \). But, we can’t just arbitrarily introduce a factor of \( \sqrt{2} \). **We need to use this factor in a fraction over itself** (to multiply by the number 1, which is ok).

- In general, if \( a \geq 0 \), then \( \sqrt{a} \cdot \sqrt{a} = a \) (This follows from the Product Property of Radicals and the fact that \( \sqrt{x^2} = x \) when \( x \) is non-negative.)

So, to finish the example, \( \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} \) in effect, multiplying by 1 in the form \( \frac{\sqrt{2}}{\sqrt{2}} \)

\[
= \frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} \quad \text{fraction multiplication}
\]

\[
= \frac{\sqrt{2}}{2} \quad \text{since } \sqrt{2} \cdot \sqrt{2} = 2
\]

This example now meets all three conditions for simplest form.

**Example 7** Simplify. Assume that all variables represent non-negative real numbers.

a. \( \sqrt[3]{\frac{5}{3}} \)  

Solution:  

\[
\sqrt[3]{\frac{5}{3}} = \frac{\sqrt[3]{5}}{\sqrt[3]{3}} \quad \text{by the Quotient Property of Radicals}
\]

\[
= \frac{\sqrt[3]{5} \cdot \sqrt[3]{3}}{\sqrt[3]{3} \cdot \sqrt[3]{3}} = \frac{\sqrt[3]{15}}{3} \quad \text{by the Product Property of Radicals and } \sqrt{x^2} = x \text{ for non-negative } x
\]

b. \( \sqrt[11]{\frac{2x}{11}} \)

Solution:  

\[
\sqrt[11]{\frac{2x}{11}} = \frac{\sqrt[11]{2x}}{\sqrt[11]{11}} \quad \text{by the Quotient Property of Radicals}
\]

\[
= \frac{\sqrt[11]{2x} \cdot \sqrt[11]{11}}{\sqrt[11]{11} \cdot \sqrt[11]{11}} = \frac{\sqrt[11]{22x}}{11} \quad \text{by the Product Property of Radicals and } \sqrt{x^2} = x \text{ for non-negative } x
\]
Simplify. Assume that all variables represent non-negative real numbers.

24. \( \sqrt[3]{\frac{1}{3}} \)
25. \( \sqrt[3]{\frac{4}{3}} \)
26. \( \sqrt[3]{\frac{3y}{7}} \)

### Exercises for the Radicals Packet, Part 2

#### a and b, and Part 1 packet:
Evaluate if possible.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \sqrt{400} )</td>
<td>5</td>
<td>( \sqrt{-\frac{1}{25}} )</td>
</tr>
<tr>
<td>2</td>
<td>( -\sqrt{100} )</td>
<td>6</td>
<td>( \sqrt[4]{27} )</td>
</tr>
<tr>
<td>3</td>
<td>( \sqrt{-100} )</td>
<td>7</td>
<td>( \sqrt[4]{64} )</td>
</tr>
<tr>
<td>4</td>
<td>( -\sqrt{-\frac{1}{25}} )</td>
<td>8</td>
<td>( \sqrt{-27} )</td>
</tr>
</tbody>
</table>

#### a and b, and Part 1 packet:
Which of the following roots are rational numbers and which are irrational numbers?

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<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>15</td>
<td>( \sqrt{19} )</td>
<td>17</td>
<td>( \sqrt[3]{9} )</td>
</tr>
<tr>
<td>16</td>
<td>( \sqrt{36} )</td>
<td>18</td>
<td>( \sqrt[3]{8} )</td>
</tr>
</tbody>
</table>

#### b
Evaluate each of the following expressions.

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<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>( \sqrt[3]{5^3} )</td>
<td>26</td>
<td>( \sqrt[3]{4^3} )</td>
</tr>
<tr>
<td>25</td>
<td>( \sqrt{-5^2} )</td>
<td>27</td>
<td>( \sqrt[4]{(-3)^4} )</td>
</tr>
</tbody>
</table>

#### a and b:
Find the two expressions that are equivalent. (Hint: evaluate each expression if possible first.)

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<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>32</td>
<td>( -\sqrt{16} ), ( -\sqrt{16} ), ( -4 )</td>
<td>34</td>
<td>( \sqrt{-125} ), ( -\sqrt[5]{125} ), ( -5 )</td>
</tr>
<tr>
<td>33</td>
<td>( -\sqrt{25} ), ( -5 ), ( \sqrt{-25} )</td>
<td>35</td>
<td>( \sqrt{-32} ), ( -\sqrt[3]{32} ), ( -2 )</td>
</tr>
</tbody>
</table>

#### c
Use the Product Property of Radicals to simplify the following expressions. Assume that all variables represent non-negative real numbers.

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<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>38</td>
<td>( \sqrt{18} )</td>
<td>40</td>
<td>( \sqrt{125} )</td>
</tr>
<tr>
<td>39</td>
<td>( \sqrt{80} )</td>
<td>41</td>
<td>( \sqrt{5x^2} )</td>
</tr>
</tbody>
</table>

#### C
Use the Quotient Property of Radicals to simplify the following expressions.

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<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>( \sqrt[2]{\frac{4}{25}} )</td>
<td>47</td>
<td>( \sqrt[9]{5} )</td>
</tr>
<tr>
<td>46</td>
<td>( \sqrt[4]{\frac{3}{4}} )</td>
<td>48</td>
<td>( \sqrt[49]{64} )</td>
</tr>
</tbody>
</table>
Use the Quotient and Product Properties of Radicals to simplify the following expressions. Assume that all variables represent non-negative real numbers.

49. \[ \frac{8a^2}{25} \]
50. \[ \frac{1}{\sqrt{5}} \]
51. \[ \frac{3}{\sqrt{2}} \]
52. \[ \frac{2x}{\sqrt{7}} \]
53. \[ \frac{2x^2}{\sqrt{3}} \]
54. \[ \frac{8s^3}{\sqrt{7}} \]

**Combining Concepts.** This symbol means you need to write the answer in complete sentences.

55. Explain why \( \sqrt{x^2} \) does not equal \( x \) for all real numbers. Give an example to support your reasoning.
56. Why is the use of absolute value not required for the expression \( \sqrt{x^n} \) when \( n \) is odd?
57. Does the \( n^{th} \) root of \( x^2 \) always exist? Why or why not?
58. Why will writing \( \sqrt{50} = \sqrt{10} \cdot 5 \) not help in writing the expression in simplified form?

Evaluate each of the following expressions. Assume that all variables represent any real number.

59. \( \frac{999}{10} (2a+b)^{999} \)
60. \( \sqrt[414]{(a+b)^414} \)

Decide whether each of the following is already in simplest form. If not, explain what needs to be done.

61. \( \sqrt{10mn} \)
62. \( \sqrt{18ab} \)
63. \( \frac{\sqrt{98x^2y}}{7x} \)
64. \( \sqrt{\frac{6xy}{3x}} \)
Practice Exercises: 1. inverse; 3. 0; 5. 12 and -12; 7. radical sign or radical; 9. 8; 11. 0; 13. ¾; 15. ≈5.385;
17. Since $8^2 = 64$, and $\sqrt{64}$ asks just for the positive root, $\sqrt{64} = 8$ is a true statement.
19. Since 43 is not a perfect square, $\sqrt{43}$ must be irrational.
   Alternately, $\sqrt{43} \approx 6.557438524302$, which is a non-terminating, non-repeating decimal, so $\sqrt{43}$ must be irrational.

Exercises: 1. 12 and -12; 3. -3; 5. 6 and 7; 7. $\approx 33.166$;
9. ($\approx 2.646$, a non-terminating, non-repeating decimal, or, 7 isn’t a perfect square), irrational;

Practice Exercises:
1. not real 11. 7 19. $2c\sqrt{2c}$
3. $½$ 13. -1 21. $\frac{5}{4}$
5. 3 15. $3\sqrt{5}$ 23. $\frac{5\sqrt{2}}{7}$
7. not real 17. $9\sqrt{2}$
9. 7

Exercises:
1. 20 5. not real 9. not real 13. 10
3. not real 7. 3 11. 2

15. $= 4.358898944$ or 19 is not a perfect square, irrational
17. $= 2.080083823$ or 9 is not a perfect cube, irrational
19. $= 2$, rational
21. $= .755928946$ or $4/7$ is not a perfect square, irrational
23. $= -3$, rational

25. 5 27. 3 29. -5 31. -2

in # 33-37, these results are in the order presented:
33. -5, -5, not real ⇒ the first two are equivalent
35. -2, -2, 2 ⇒ the first two are equivalent
37. 100, 100, ≈46.41588834 ⇒ the first two are equivalent

39. $4\sqrt{5}$ 41. $x\sqrt{5}$ 43. $3a^2\sqrt{6a}$
45. $\frac{2}{5}$ 47. $\frac{\sqrt{5}}{3}$

49. $\frac{2a\sqrt{2}}{5}$ 51. $\frac{\sqrt{6}}{2}$ 53. $\frac{x\sqrt{6}}{3}$

55. ✷ Answers vary 57. ✷ Answers vary
59. $2a + b$ 60. $|a + b|$