Interval Notation represents an entire continuous segment of the real number line by using grouping symbol(s) (parentheses or brackets), and just the endpoints of the entire set separated by a comma.

Recall, in set-builder notation, the set including all real numbers (including all integers, fractions, decimals, square roots, etc.) from \( a \) to \( b \) (where \( a \leq b \)) is written as \( \{ x \mid a \leq x \leq b \} \), and has the number line graph (with closed ends)

\[
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\]

In interval notation, a bracket is used on any included endpoint, and only the endpoints are listed, but the set will still include all numbers between. Another restriction is that the endpoints must always be written in least-to-greatest order. Here is the interval notation for the same set above: \([a, b]\).

(Closed): All three of the representations above (set-builder notation, graph, interval notation) are for the same set and they each contain all the real numbers between \( a \) and \( b \), plus the endpoints \( a \) and \( b \) themselves (i.e., they contain all real numbers from \( a \) to \( b \), inclusive).

This set is called closed, because it includes both its endpoints.

(Open): In set-builder notation, the set including all real numbers from \( a \) to \( b \) (where \( a < b \)) EXCEPT the endpoints is written as \( \{ x \mid a < x < b \} \), and has the number line graph (with open ends)

\[
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\]

In interval notation, a parenthesis is used on any excluded endpoint. But here again, only the endpoints are listed. Here is the interval notation for the same set shown just above: \((a, b)\).

All three of the representations above (set-builder notation, graph, interval notation) are for the same set and they each contain all the real numbers between \( a \) and \( b \), EXCEPT NOT the endpoints \( a \) and \( b \) themselves (i.e., they contain all real numbers from \( a \) to \( b \), exclusive).

This set is called open, because it excludes both its endpoints.

If a set contains one of its endpoints, but not the other, we call that half-closed, as in the next two cases.

(Half-Closed (at left –end)): In set-builder notation, the set including all real numbers from \( a \) to \( b \) (where \( a < b \)), except not the point \( b \) itself, is written as \( \{ x \mid a \leq x < b \} \), and has the number line graph (with the left end closed and right end open)

\[
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\]

In the interval notation above, the left end will have a bracket, while the right end gets a parenthesis. Here is the interval notation for that set: \([a, b)\).
All three of the representations above (set-builder notation, graph, interval notation) are for the same set and they each contain all the real numbers between $a$ and $b$, including the endpoint $a$ but excluding the endpoint $b$. Again, this set is **half-closed**.

(Closed at right-end): In **set-builder notation**, the set including all real numbers from $a$ to $b$ (where $a < b$), except not the point $a$ itself, is written as \{x | a < x ≤ b\}, and has the number line graph (with the left end open and right end closed)

\[ a \quad b \]

In interval notation here, the left end will have a **parenthesis**, while the right end gets a **bracket**. The interval notation for this set is: $(a, b]$.

All three of the representations above (set-builder notation, graph, interval notation) are for the same set and each contains all the real numbers between $a$ and $b$, excluding the endpoint $a$ but including the endpoint $b$. Again, this set is **half-closed**.

**Example 1**: State each of the following sets using interval notation. Also state whether the set is closed, open, or half-closed.

A) \{x | –10 < x < 3\}  B) \{x | 3 < x ≤ 5\}

C) \{x | –1 < x < –2\}  D) \{x | –2 ≤ x ≤ –2\}

**Solutions**:

A) \{x | –10 < x < 3\}
Since the constraints are –10 < x < 3, and we know –10 < 3 (meaning this set does have numbers in it), we know that we need to use parentheses on both ends (since both inequality arrows are strictly “is less than,” with no “or equal to” bar). This makes the interval notation for this set \((-10, 3)\). Also, since neither endpoint is included, the set is open.

B) \{x | 3 < x ≤ 5\}
Since 3 < x ≤ 5, and we know 3 ≤ 5, we know that we need to use a parenthesis on the left end and a bracket on the right end (since the left inequality arrow is strictly “is less than,” with no “or equal to” bar, but the right inequality is “is less than or equal to”). This makes the interval notation for this set \((3, 5]\). Also, since one endpoint is included, the set is half-closed.

C) \{x | –1 < x < –2\} (Careful on this one.)
Since –1 < x < –2 is a **nonsense statement** (there aren’t any numbers $x > –1$ and $x < –2$ at the same time, which is what is says), the graph looks like this:

\[ \]

(no shading, since $x$ is greater than 1 AND less than –2 at the same time)

Since there are no numbers right of –1 that are also left of –2, this set is empty!

The interval notation here is the same as the final correct set-builder notation: \(∅\).
C) (continued) A cautionary note: though tempting, \((-1, -2)\) is not the interval notation here, because these endpoints are not in least-to-greatest order. If written that way, that interval is empty. Because of this unusual case, it’s neither open nor closed (those have to have at least a point in them).

D) \(\{x | -2 \leq x \leq -2\}\)

Here we know that we need to use brackets on both ends (since both inequality arrows are “is less than or equal to” this time). This makes the interval notation for this set \([-2, -2]\). Also, since “both” endpoints are included, the set is closed. NOTE: This set only has one element in it, the number \(-2\), and so it could be written much more concisely using plain set notation as \(-2\).

Example 2: State the each of the following sets graphed here using interval notation.

A) \([-3/5, 7/2]\)

Since the graph has brackets on both ends, that means this set is closed (meaning this set does have the endpoints in it). We also know that we need to use brackets on both ends in the interval notation. Since the endpoints are \(-3/5\) on the left and \(7/2\) on the right, and we want to include all the points between, the interval notation for this set is \([-3/5, 7/2]\).

B) \([5.5, 100.2]\)

Here, the left end is bracketed and the right end has a parenthesis, making this set half-closed. So the left end will be a bracket and the right end a parenthesis on the interval notation as well. Here the left end is the number \(5.5\), with the right being the number \(100.2\), and we want to include all numbers between again. So the interval notation is \([5.5, 100.2]\).

C) \([-2\), openended to \(-2]\)

The interval notation starts and stops with \(-2\), so that is both endpoints, and it becomes \([-2, -2]\), but this is not the standard way to represent single points. Since there is only one number in the set, it’s easier to list in set notation as \{-2\}.
D)

On this graph, the left side has a finite endpoint, \(-2\), but the right end never stops. Recall that at the ends, if a set goes on forever like that, it is said to go towards infinity. So we can think of this graph as having this symbol for infinity, \(\infty\), as it’s right “end point.” (Since it’s not a true number, we can’t really call it an end point, hence the quotation marks.)

With that new addition to the picture, we can easily see that the interval notation is \((-2, \infty)\). Note, even if the other endpoint is bracketed, infinity always gets written with a parenthesis because it’s it’s not finite (not one fixed number as an endpoint). Since the left endpoint is not included here, and the right end is infinity, this set is open.

NOTE: If there is only one fixed (finite) end to a set, and it is included (written with a bracket), then the set is still considered closed.

Example 3: Graph each set on a number line. State each of the following sets using interval notation. Also state whether the set is closed, open, or half-closed.

A) \(\{x | x < 3\}\)  

Here, since \(x\) is left of 3, we know the graph ends at positive 3, with all the points in the set running left of that end.

With that and if we put our imagined infinity at the left end (here, since it’s on the left, it’s written as negative infinity, \(-\infty\)),

we can see that the interval notation is \([-\infty, 3)\). Remember, infinity is always parenthesized, and here \(x\) is less than (but not equal to) 3, so 3 is not in the set and is also parenthesized. As the 3 is excluded from the set, this set is considered \(\text{open}\).

B) \(\{x | x \leq 5\}\)

Here, since \(x\) is left of or equal to 5, we know the graph ends at positive 5 with all the points in the set running left of that end. But here, since \(x\) can be 5 itself, the endpoint is included, and so it gets a bracket. With that and if we put our imagined infinity at the left end again (left is still written as negative infinity, \(-\infty\)),

we can see that the interval notation is \([-\infty, 5]\).
B) (continued) Infinity is *still* parenthesized, even though the 5 gets a bracket. As the 5 is included in the set, *even though the other end is infinite*, this set is considered [closed].

C) \( \{x | -1 < x \} \)

On this example, it may help to rewrite the inequality with the \( x \) on the left side (easier to read). If \(-1 < x \), then \( x > -1 \). We can now see that the graph of this set would start at \(-1 \) and run right of that, but \(-1 \) would not be in the set.

So, this graph never stops toward the right here. This would put our imaginary infinity at the right end here, and we can see that the interval notation becomes \((-1, \infty)\) this time. Here, the \(-1 \) is not included, and so this set is [open].

D) \( \{x | -2 \leq x \leq -1 \} \)

Here, the graph will include both the endpoints, and both are known (finite). This tells us that the interval notation will include the endpoints as well, and so they will be in brackets: \([-2, -1]\). Since both ends here are included in the set, this set is [closed].

From the last example, we can see that at times, there is not a known end-point value in a set. When this happens, we will state the end as infinite using the symbol for infinity, \( \infty \). Because infinity is not a single known value, but the indication that there isn’t one specific value, when used in interval notation we always put a parenthesis around any infinite end of a set.

So, that means that we have just eight ways to represent all the possible infinite sets, when they are continuous (no gaps). I have summarize them in the following table. In correct notation, the value \( a \) must be less than (or equal to, if \( \leq \) is used) the value \( b \), for the set to contain any numbers.
<table>
<thead>
<tr>
<th>Set-builder notation</th>
<th>Graph</th>
<th>Interval Notation</th>
<th>Open/Closed/Half-Closed</th>
</tr>
</thead>
<tbody>
<tr>
<td>{x \mid a \leq x \leq b}</td>
<td><img src="image1" alt="Graph" /></td>
<td>[a, b]</td>
<td>closed</td>
</tr>
<tr>
<td>{x \mid a &lt; x &lt; b}</td>
<td><img src="image2" alt="Graph" /></td>
<td>(a, b)</td>
<td>open</td>
</tr>
<tr>
<td>{x \mid a &lt; x \leq b}</td>
<td><img src="image3" alt="Graph" /></td>
<td>((a, b])</td>
<td>half-closed</td>
</tr>
<tr>
<td>{x \mid a \leq x &lt; b}</td>
<td><img src="image4" alt="Graph" /></td>
<td>([a, b)]</td>
<td>half-closed</td>
</tr>
<tr>
<td>{x \mid x \leq b}</td>
<td><img src="image5" alt="Graph" /></td>
<td>((-\infty, b])</td>
<td>closed</td>
</tr>
<tr>
<td>{x \mid x &lt; b}</td>
<td><img src="image6" alt="Graph" /></td>
<td>((-\infty, b))</td>
<td>open</td>
</tr>
<tr>
<td>{x \mid x \geq a}</td>
<td><img src="image7" alt="Graph" /></td>
<td>([a, \infty))</td>
<td>closed</td>
</tr>
<tr>
<td>{x \mid x &gt; a}</td>
<td><img src="image8" alt="Graph" /></td>
<td>((a, \infty))</td>
<td>open</td>
</tr>
</tbody>
</table>

### Exercise Set

For each of the following, A) graph the set on a real number line, B) write the interval notation and C) state whether the set is open, closed, or half-closed.

1. \{x \mid 10 < x < 13\}  
2. \{x \mid x > -10\}  
3. \{x \mid -1 \leq x \leq 5\}  
4. \{x \mid -1 \leq x < 5\}  
5. \{x \mid x < -1\}  
6. \{x \mid x \geq 5\}  
7. \{x \mid -3 < x \leq -1\}  

Write the interval notation for each set graphed below.

9. ![Graph](image9)  
10. ![Graph](image10)  
11. ![Graph](image11)  
12. ![Graph](image12)  

For the following, state the interval notation. If empty, use the empty set symbol: \(\emptyset\), if only one value is in the set, state in set notation.

13. \{x \mid -3 < x < -3\}  
14. \{x \mid -3 < x < -4\}  
15. \{x \mid -3 \leq x \leq -3\}