I. Limits

A. Informal Definition of a Limit

Jessica has a new boyfriend. His name is Ben. Jessica’s father is not fond of Ben. He imposes a midnight curfew on Jessica and warns her that each time she is late, he will cut her weekly allowance in half. Currently Jessica’s weekly allowance is $100.

According to the rule, the first time Jessica is late her allowance will drop from $100 to $50. The second time she is late, her allowance will drop from $50 to $25. The third time she is late, her allowance will drop from $25 to $12.50, and so on.

According to this model, as the number of times she is late increases, her allowance will get closer and closer to zero. Theoretically (assuming we can make a coin for any denomination, no matter how small), it will never actually reach zero. In mathematical terms, we can say that the limit of her allowance is zero.

The limit of a function is an output value (L) the function gets closer and closer to as the input value gets closer and closer to some number (a).

B. Limit Notation

An arrow, →, stands for the word “approaches”. Thus the statement, “as x approaches 3, y approaches 7” can be written “as x → 3, y → 7”.

\[ \lim_{x \to 3} (2x + 1) = 7 \] is read as “the limit, as x approaches 3, of 2x + 1, is 7.”

As \( x \to 3 \) is read “as x approaches 3 from the left”. (i.e., \( x < 3 \); for example: \( x = 2.9, x = 2.99 \))

As \( x \to 3^+ \) is read “as x approaches 3 from the right”. (i.e., \( x > 3 \); for example: \( x = 3.1, x = 3.01 \))

\[ \lim_{x \to a^-} f(x) = L \] is read as “the limit, as x approaches a from the left, of f of x is L”. It is called the left-hand limit.

\[ \lim_{x \to a^+} f(x) = L \] is read as “the limit, as x approaches a from the right, of f of x is L”. It is called the right-hand limit.

C. Formal Definition of a Limit

As x approaches a, the limit of f(x) is L, written \( \lim_{x \to a} f(x) = L \), if all values of f(x) are close to L for values of x that are sufficiently close, but not equal, to a. The limit L must be a unique real number.

D. Theorem on One-Sided Limits

As x approaches a, the limit of f(x) is L if the left-hand and right-hand limits both exist and both are L. That is,

\[ \lim_{x \to a^-} f(x) = L \] and \( \lim_{x \to a^+} f(x) = L \), then \( \lim_{x \to a} f(x) = L \).

Note: The converse of this theorem is also true. If the overall limit is L, then the left-hand and right-hand limits both exist and both are L.
II. Finding Limits

A. A Numerical Approach: The Table Method

To find \( \lim_{x \to a} f(x) \), \( \lim_{x \to a^-} f(x) \), and / or \( \lim_{x \to a^+} f(x) \):

1. Enter the equation into your calculator at \( y_1 = \) and adjust the window so you can see the key characteristics of the graph.

2. If you have not done so already, go to the TBLSET window under 2nd WINDOW and set Indpnt on Ask. Leave Depend on Auto. Go to the TABLE at 2nd GRAPH.

3. To find \( \lim_{x \to a^-} f(x) \), enter in the table three x-values less than a (i.e., \( a - .1 \), \( a - .01 \), \( a - .001 \)). Observe what happens to the y-values as the x-values get closer to a. If they seem to be getting closer to a specific y-value, record that number as the left limit of \( f(x) \).

To find \( \lim_{x \to a^+} f(x) \), enter in the table three x-values greater than a (i.e., \( a + .1 \), \( a + .01 \), \( a + .001 \)). Observe what happens to the y-values as the x-values get closer to a. If they seem to be getting closer to a specific y-value, record that number as the right limit of \( f(x) \).

If \( \lim_{x \to a^-} f(x) = L \) and \( \lim_{x \to a^+} f(x) = L \), then \( \lim_{x \to a} f(x) = L \).

If \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \) are not the same, then \( \lim_{x \to a} f(x) = \text{DNE} \).

Example 1

Consider the function given by \( f(x) = \begin{cases} x - 2, & \text{for } x \leq 3 \\ x - 1, & \text{for } x > 3 \end{cases} \). Find the following limits.

When necessary, state that the limit does not exist.

a. Find \( \lim_{x \to 3^-} f(x) \).

<table>
<thead>
<tr>
<th>( x \to 3^- )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.9</td>
<td>.9</td>
</tr>
<tr>
<td>2.99</td>
<td>.99</td>
</tr>
<tr>
<td>2.999</td>
<td>.999</td>
</tr>
</tbody>
</table>

Thus \( \lim_{x \to 3^-} f(x) = 1 \).

b. Find \( \lim_{x \to 3^+} f(x) \).

<table>
<thead>
<tr>
<th>( x \to 3^+ )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>2.1</td>
</tr>
<tr>
<td>3.01</td>
<td>2.01</td>
</tr>
</tbody>
</table>

Thus \( \lim_{x \to 3^+} f(x) = 2 \).

c. Find \( \lim_{x \to 3} f(x) \).

<table>
<thead>
<tr>
<th>( x \to 3 )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.001</td>
<td>2.001</td>
</tr>
</tbody>
</table>

Since the left and right limits are not the same, \( \lim_{x \to 3} f(x) \) does not exist (DNE).

B. A Graphical Approach: The Wall Method

To find \( \lim_{x \to a^-} f(x) \), \( \lim_{x \to a^+} f(x) \), and / or \( \lim_{x \to a} f(x) \):

1. If you are given a graph, skip to step 2.

If you are not given a graph, enter the equation into your calculator at \( y_1 = \) and adjust the window so you can see the key characteristics of the graph.
Hint: To enter the equation of a piecewise function such as
\[ f(x) = \begin{cases} 
2x + 4, & x \leq 1 \\
7 - x, & x > 1 
\end{cases} \]
at \( y_1 \), enter \((2x + 4)(x \leq 1) + (7 - x)(x > 1)\).
Inequality symbols such as \( \leq \) and \( > \) are under the TEST menu at 2nd MATH.

2. Draw a vertical line through \( x = a \). This line is called the wall.
To find \( \lim_{x \to a^-} f(x) \), on the left side of \( x = a \), follow the curve from left to right until you hit the wall. Mark the location with an \( x \).
To find \( \lim_{x \to a^+} f(x) \), on the right side of \( x = a \), follow the curve from right to left until you hit the wall. Mark the location with an \( x \).
If the two exes overlap, \( \lim_{x \to a} f(x) \) is the y-value of the point in the center of the x.
If the two exes do not overlap, the overall limit does not exist ( \( \lim_{x \to a} f(x) = \text{DNE} \) ).

**Example 2** Use the adjacent graph to find each limit. When necessary, state that the limit does not exist.

a. \( \lim_{x \to -3} F(x) \)

First draw a vertical line at \( x = -3 \). Following the curve from left to right, we hit the wall at \((-3, 5)\). Thus \( \lim_{x \to -3} F(x) = 5 \).
Following the curve from right to left, we hit the wall at \((-3, 5)\). Thus \( \lim_{x \to -3^+} F(x) = 5 \).
Since the left and right limits are both 5, \( \lim_{x \to -3} F(x) = 5 \).

b. \( \lim_{x \to -2} F(x) \)

First draw a vertical line at \( x = -2 \). Following the curve from left to right, we hit the wall at \((-2, 4)\). Thus \( \lim_{x \to -2} F(x) = 4 \).

(c) \( \lim_{x \to -2^+} F(x) \)

Following the curve from right to left, we hit the wall at \((-2, 2)\). Thus \( \lim_{x \to -2^+} F(x) = 2 \).

(d) \( \lim_{x \to -2} F(x) \)

Since the left and right limits are not the same, \( \lim_{x \to -2} F(x) \) does not exist (DNE).

e. \( \lim_{x \to 4} F(x) \)

First draw a vertical line at \( x = 4 \). Following the curve from left to right, we hit the wall at \((4, 2)\). Following the curve from right to left, we again hit the wall at \((4, 2)\). Thus \( \lim_{x \to 4^-} F(x) = 2 \); \( \lim_{x \to 4^+} F(x) = 2 \); and \( \lim_{x \to 4} F(x) = 2 \).

C. **Observations**

Sometimes \( \lim_{x \to a} f(x) = f(a) \), but not always. See example 1 and text example 3 part a.
It is possible for \( \lim_{x \to a} f(x) \) to exist even when \( f(a) \) does not exist. See example 2 part e and text example 1.
III. Limits Involving Infinity

A. Infinite Limits

When considering \( \lim_{x \to a^-} f(x) \) or \( \lim_{x \to a^+} f(x) \), it may happen that \( f(x) \) increases or decreases without bound (that is, becomes infinitely large or infinitely small) as \( x \) approaches \( a \). In such instances, the limit does not exist in the usual sense. However, we often describe this situation by writing \( \lim_{x \to a^-} f(x) = \infty \) or \( \lim_{x \to a^+} f(x) = -\infty \).

If \( x \) approaches \( a \) from the left, the y-values decrease without bound, we say \( \lim_{x \to a^-} f(x) = -\infty \).

If \( x \) approaches \( a \) from the right, the y-values increase without bound, we say \( \lim_{x \to a^+} f(x) = \infty \).

B. Limits at Infinity

Sometimes we are concerned with the behavior of a function \( f \) as the magnitude of the input variable increases or decreases without bound (that is, the magnitude becomes infinitely large or infinitely small). Such limits are called limits at infinity and are written as \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \). The notation \( x \to \infty \) can be read “as \( x \) increases without bound” or “as \( x \) approaches infinity.” Similarly, \( x \to -\infty \) can be read “as \( x \) decreases without bound” or “as \( x \) approaches negative infinity.”

These limits are approached from one side only:

from the left if approaching positive infinity OR
from the right if approaching negative infinity.

Example 3

Use the adjacent graph to find the following limits. When necessary, state that the limit does not exist.

a. \( \lim_{x \to -3} f(x) \)

As \( x \) approaches -3 from the left, the y-values increase without bound, so we can say \( \lim_{x \to -3^-} f(x) = \infty \).

As \( x \) approaches -3 from the right, the y-values decrease without bound, so we can say \( \lim_{x \to -3^+} f(x) = -\infty \).

Since the left limit and the right limit are not the same, \( \lim_{x \to -3} f(x) \) does not exist (DNE).

b. \( \lim_{x \to \infty} f(x) \)

As \( x \) approaches infinity from the left, the y-values get closer and closer to 1. Therefore, \( \lim_{x \to \infty} f(x) = 1 \).
Example 4  
Graph the function \( g(x) = \frac{1}{x - 3} + 2 \), then find the specified limits. When necessary, state that the limit does not exist.

a. \( \lim_{x \to \infty} g(x) \)

As \( x \) approaches infinity from the left, the \( y \)-values get closer and closer to 2.
Thus \( \lim_{x \to \infty} g(x) = 2 \).

b. \( \lim_{x \to 3} g(x) \)

As \( x \) values approach 3 from the left, the \( y \)-values decrease without bound. Thus \( \lim_{x \to 3^-} g(x) = -\infty \).
As \( x \) values approach 3 from the right, the \( y \)-values increase without bound. Thus \( \lim_{x \to 3^+} g(x) = \infty \).
Since the left and right limits are not the same, \( \lim_{x \to 3} g(x) \) does not exist (DNE).

IV. Applications

Example 5  
Population Growth

In a certain habitat, the deer population (in hundreds) as a function of time (in years) is given in the adjacent graph of \( p \).

Use the graph to find the following limits.

a. \( \lim_{t \to 1.75^-} p(t) \)

As \( t \) approaches 1.75 from the left the deer population is approaching 12 hundred. Therefore, \( \lim_{t \to 1.75^-} p(t) = 1200 \).

b. \( \lim_{t \to 1.75^+} p(t) \)

As \( t \) approaches 1.75 from the right the deer population is approaching 13 hundred. Therefore, \( \lim_{t \to 1.75^+} p(t) = 1300 \).

c. \( \lim_{t \to 1.75} p(t) \)

Since the left and right limits are not the same, \( \lim_{t \to 1.75} p(t) \) does not exist (DNE).