Section 1.1 Angles

I. Basic Terminology

A. Two distinct points, A and B, determine a line called line \( AB \).

\[ A \quad B \]

B. Line segment \( AB \) is the portion of the line between A and B, including points A and B themselves.

\[ A \quad B \]

C. Ray \( AB \) is the portion of line AB that starts at A and continues through, and on past, B. Point A is called the endpoint of the ray.

\[ A \quad B \]

D. An angle is formed by rotating a ray around its endpoint.

E. The ray in its initial position is called the initial side of the angle.

F. The ray in its location after the rotation is called the terminal side of the angle.

G. The endpoint of the ray is called the vertex of the angle.

\[ \text{Terminal side} \quad \text{Vertex A} \quad \text{Initial side} \]

H. If the rotation of the terminal side is counterclockwise, the angle is positive.

I. If the rotation of the terminal side is clockwise, the angle is negative.
II. Degree Measure
   A. One of the most common units for measuring angles is the degree which is represented by the ° symbol.
   B. One degree, written 1°, is \(\frac{1}{360}\) of a complete rotation.

III. Types of Angles
   A. A complete rotation of a ray gives an angle whose measure is 360°.
   B. An angle measuring between 0° and 90° is called an acute angle.
   C. An angle measuring exactly 90° is a right angle. It represents \(\frac{1}{4}\) of a full rotation.
   D. An angle measuring between 90° and 180° is called an obtuse angle.
   E. An angle measuring exactly 180° is a straight angle. It represents \(\frac{1}{2}\) of a full rotation.

Note: We often use Greek letters such as theta (\(\theta\)), alpha (\(\alpha\)), beta (\(\beta\)), and gamma (\(\gamma\)) to name angles.

IV. Complementary & Supplementary Angles
   A. If the sum of the measures of two positive angles is 90°, the angles are called complementary.
   B. If the sum of the measures of two positive angles is 180°, the angles are called supplementary.

Example 1 Find the complement and the supplement of the angle with measure 18°. [#8]

Complement: \(90° - 18° = 72°\)

Supplement: \(180° - 18° = 162°\) or \(90° + 72° = 162°\)
V. Degrees, Minutes, Seconds (DMS)

A. One minute (′) is \( \frac{1}{60} \) of a degree and 60 minutes equals one degree.

\[ 1′ = \frac{1°}{60} \quad \text{and} \quad 60′ = 1° \]

B. One second (″) is \( \frac{1}{60} \) of a minute and \( \frac{1}{3600} \) of a degree. 60 seconds equals one minute and 3600 seconds equals 60 minutes which equals one degree.

\[ 1″ = \frac{1′}{60} = \frac{1°}{3600} \quad \text{and} \quad 60″ = 1′ \quad \text{and} \quad 3600″ = 60′ = 1° \]

C. 23° 14′ 51″ represents 23 degrees, 14 minutes, and 51 seconds.

D. 90° can be written as 89° 59′ 60″ and 180° can be written as 179° 59′ 60″.

Example 2 Perform the calculation: \( 90° - 36° 18′ 47″ \). (#30)

\[
\begin{align*}
89° & \ 59′ \ 60″ \\
-36° & \ 18′ \ 47″ \\
53° \ & \ 41′ \ 13″
\end{align*}
\]

VI. Converting between Degrees, Minutes, and Seconds (DMS) and Decimal Degrees (DD)

A. DMS to DD

Example 3 Convert the angle measure of 34° 51′ 35″ to decimal degrees. Round to the nearest thousandth of a degree. (#34)

\[
Manually:\quad 34 + \frac{51}{60} + \frac{35}{3600} = 34.85722222222222° \approx 34.860°
\]

Using a Calculator:

TI-83 / 84 Keystrokes: 34 2nd MATRIX 1 51 2nd MATRIX 2 35 ALPHA + ENTER 34° 51′ 35″ ENTER 34.85722222222222° \approx 34.860°

TI-82 Keystrokes: 34 2nd MATRIX 2 51 2nd MATRIX 2 35 2nd MATRIX 2 ENTER 34° 51′ 35″ ENTER 34.85722222222222° \approx 34.860°
B. DD to DMS

**Example 4**  Convert the angle measure of 59.0854° to degrees, minutes, and seconds. (#38)

*Manually:* \[59.0854° - 59° = 0.0854° \rightarrow 0.0854° \times \frac{60'}{1°} = 5.124' \rightarrow 5.124' - 5' = 0.124' \rightarrow 0.124' \times \frac{60''}{1'} = 7.44'' \rightarrow 59° 5' 7''\]

*Using a Calculator:* 59.0854° 2nd MATRIX 4 ENTER 59° 5' 7.44'' \rightarrow 59° 5' 7''

VII. **Standard Position**

A. An angle is in **standard position** if its vertex is at the origin and its initial side is along the positive x-axis. The angle is said to lie in the quadrant in which the terminal side is located.

B. Angles in standard position having their terminal sides along the x-axis or y-axis are called **quadrantal angles**. These include 0°, 90°, 180°, 270°, and 360°.

VIII. **Coterminal Angles**

A complete rotation of a ray results in an angle measuring 360°. By continuing the counterclockwise rotation, angles larger than 360° can be produced. By rotating in a clockwise direction, negative angles can be produced. Angles 53° and 413° and -307° have the same initial side and the same terminal side, but different amounts of rotation. Such angles are called **coterminal angles**.
The measures of coterminal angles differ by a multiple of $360^\circ$. Thus any angle coterminal with angle $A$ can be written in the form $A^\circ + n(360^\circ)$ or $A^\circ - n(360^\circ)$.

**Example 5** Find the angles of smallest possible positive measure coterminal with each angle.

a) $699^\circ$ (#50)  
b) $-203^\circ$ (#48)

a. $699^\circ - 360^\circ = 339^\circ$

b. $-203^\circ + 360^\circ = 360^\circ - 203^\circ = 157^\circ$

**IX. Application**

**Example 6** An airplane propeller rotates 1000 times per minute. Find the number of degrees that a point on the edge of the propeller will rotate in 1 second. (#78)

$$\frac{1000 \text{ rotations}}{1 \text{ minute}} \times \frac{360^\circ}{1 \text{ rotation}} \times \frac{1 \text{ minute}}{60 \text{ seconds}} = 6000^\circ \text{ per second}$$

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**Section 1.3 Trigonometric Functions**

I. **Definition of Trigonometric Functions**

Let $(x, y)$ be a point other than the origin on the terminal side of an angle $\theta$ in standard position.

The distance from the point to the origin is $r = \sqrt{x^2 + y^2}$.

Note: $r$ is always positive.

The six trigonometric functions of $\theta$ (sine, cosine, tangent, cosecant, secant, cotangent) are defined as follows:

\[
\begin{align*}
\sin \theta &= \frac{y}{r} \\
\cos \theta &= \frac{x}{r} \\
\tan \theta &= \frac{y}{x} \quad (x \neq 0) \\
\csc \theta &= \frac{r}{y} \quad (y \neq 0) \\
\sec \theta &= \frac{r}{x} \quad (x \neq 0) \\
\cot \theta &= \frac{x}{y} \quad (y \neq 0)
\end{align*}
\]
II. Finding Function Values of an Angle

Example 1 The terminal side of angle \( \theta \) in standard position passes through the point \((3, -4)\). Find the values of the six trigonometric functions of angle \( \theta \). (#12)

\[
x = 3 \quad y = -4 \quad r = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5
\]

\[
\sin \theta = -\frac{4}{5} \quad \cos \theta = \frac{3}{5} \quad \tan \theta = -\frac{4}{3} \\
\csc \theta = -\frac{5}{4} \quad \sec \theta = \frac{5}{3} \quad \cot \theta = -\frac{3}{4}
\]

Pythagorean Triples: 3, 4, 5 \quad 5, 12, 13 \quad 7, 24, 25 \quad 8, 15, 17 \quad 9, 40, 41

Example 2 The terminal side of angle \( \theta \) in standard position passes through the point \((-2\sqrt{3}, -2)\). Find the EXACT values of the six trigonometric functions of angle \( \theta \). (#10)

Note: When you are asked to find EXACT function values, you may not use a calculator to find the values. Your answers must be expressed in simplified radical and / or fraction form, not rounded decimal form.

\[
x = -2\sqrt{3} \quad y = -2 \quad r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{4(3) + 4} = \sqrt{16} = 4
\]

\[
\sin \theta = -\frac{2}{4} = -\frac{1}{2} \quad \cos \theta = -\frac{2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2} \quad \tan \theta = -\frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\
\csc \theta = -\frac{4}{2} = -2 \quad \sec \theta = -\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{2\sqrt{3}}{3} \quad \cot \theta = -\frac{2\sqrt{3}}{-2} = \sqrt{3}
\]

III. The Unit Circle

A unit circle is a circle with its center at the origin and a radius of 1. If we connect a point \((x, y)\) on the unit circle to the origin and drop a perpendicular from the point to the x-axis, we can create a right triangle with central angle \( \theta \), horizontal side x, vertical side y, and hypotenuse r = 1.

For this triangle, \( \cos \theta = \frac{x}{r} = \frac{x}{1} = x \) and \( \sin \theta = \frac{y}{r} = \frac{y}{1} = y \).

Generalizing, for any point \((x, y)\) on the unit circle, \( x = \cos \theta \) and \( y = \sin \theta \).
IV. Trigonometric Function Values of Quadrantal Angles

If the terminal side of a quadrantal angle lies along the x-axis, then the cosecant and cotangent are undefined. If it lies along the y-axis, then the secant and tangent are undefined.

We can use these facts, along with the four points on the unit circle associated with the quadrantal angles, to generate the following table.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Point</th>
<th>cos ( \theta ) = ( x )</th>
<th>sec ( \theta ) = ( \frac{1}{x} )</th>
<th>sin ( \theta ) = ( y )</th>
<th>csc ( \theta ) = ( \frac{1}{y} )</th>
<th>tan ( \theta ) = ( \frac{y}{x} )</th>
<th>cot ( \theta ) = ( \frac{x}{y} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°, 360°</td>
<td>(1, 0)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>undefined</td>
<td>0</td>
<td>undefined</td>
</tr>
<tr>
<td>90°</td>
<td>(0, 1)</td>
<td>0</td>
<td>undefined</td>
<td>1</td>
<td>1</td>
<td>undefined</td>
<td>0</td>
</tr>
<tr>
<td>180°</td>
<td>(-1, 0)</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>undefined</td>
<td>0</td>
<td>undefined</td>
</tr>
<tr>
<td>270°</td>
<td>(0, -1)</td>
<td>0</td>
<td>undefined</td>
<td>-1</td>
<td>-1</td>
<td>undefined</td>
<td>0</td>
</tr>
</tbody>
</table>

Note:
You should either memorize the function values of the quadrantal angles or be able to quickly generate them from the unit circle.

Example 3
The terminal side of angle \( \theta \) in standard position passes through the point (-4, 0). Find the EXACT values of the six trigonometric functions of angle \( \theta \). (#8)

At (-4, 0), angle \( \theta \) is 180° and \( x = -4 \) \( y = 0 \) \( r = \sqrt{(-4)^2 + (0)^2} = \sqrt{16} = 4 \), therefore

\[
\sin \theta = \frac{y}{r} = 0 \quad \text{and} \quad \tan \theta = \frac{y}{x} = 0
\]

\[
\csc \theta = \frac{r}{y} = \text{undefined} \quad \text{and} \quad \cot \theta = \frac{x}{y} = \text{undefined}
\]

\[
\cos \theta = \frac{x}{r} = -1 \quad \text{and} \quad \sec \theta = \frac{r}{x} = -1
\]

Example 4
Use the trigonometric function values of quadrantal angles to evaluate the following expression: \( 2 \sec 0^\circ + 4 \cot^2 90^\circ + \sin 270^\circ \) (like #38)

\[2 \sec 0^\circ + 4 \cot^2 90^\circ + \sin 270^\circ = 2(1) + 4(0)^2 + (-1) = 1\]

support work: \( 0^\circ \rightarrow (1, 0) \quad \sec 0^\circ = \frac{1}{x} = \frac{1}{1} = 1 \)

\[90^\circ \rightarrow (0, 1) \quad \cot 90^\circ = \frac{x}{y} = \frac{0}{1} = 0\]

\[270^\circ \rightarrow (0, -1) \quad \sin 270^\circ = y = -1\]
Section 1.4 Using the Definitions of the Trigonometric Functions

I. Reciprocal Identities

\[
\begin{align*}
\csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta} \\
\sin \theta &= \frac{1}{\csc \theta} & \cos \theta &= \frac{1}{\sec \theta} & \tan \theta &= \frac{1}{\cot \theta}
\end{align*}
\]

(These identities hold for any angle \( \theta \) that does not lead to a 0 denominator.)

Example 1 Use the appropriate reciprocal identity to find \( \sec \theta \) if \( \cos \theta = -\frac{\sqrt{7}}{7} \). (#6)

\[
\sec \theta = \frac{1}{\cos \theta} = \frac{1}{-\frac{\sqrt{7}}{7}} = -\sqrt{7}
\]

Note: To find the value of a reciprocal trig function (\( \csc \theta \), \( \sec \theta \), \( \cot \theta \)) on a calculator, either enter the expression as \( \frac{1}{\text{fxn} \ \theta} \) or as \( (\text{fxn} \ \theta)^{-1} \) using the reciprocal key \( [x^{-1}] \).

Do not use the inverse function keys( \( \sin^{-1} \), \( \cos^{-1} \), \( \tan^{-1} \)) to find reciprocal function values. The inverse of a function is not equal to its reciprocal. i.e., \( \text{fxn}^{-1} \ \theta \neq (\text{fxn} \ \theta)^{-1} \).

\[
\csc \theta \neq \sin^{-1} \theta \quad \sec \theta \neq \cos^{-1} \theta \quad \cot \theta \neq \tan^{-1} \theta
\]

Example 2 Use a calculator to find \( \cot 20^\circ \).

With the calculator set on degree mode, type in either \( \frac{1}{\tan 20} \) or \( (\tan 20)^{-1} \)

\[\cot 20^\circ = 2.7475\]

II. Signs of Function Values

As the adjacent diagram illustrates,

for an angle \( \theta \) which terminates in **quadrant I**, all function values are positive.

for an angle \( \theta \) which terminates in **quadrant II**, only \( \sin \theta \) and its reciprocal \( \csc \theta \) are positive.

for an angle \( \theta \) which terminates in **quadrant III**, only \( \tan \theta \) and its reciprocal \( \cot \theta \) are positive.

for an angle \( \theta \) which terminates in **quadrant IV**, only \( \cos \theta \) and its reciprocal \( \sec \theta \) are positive.

A useful mnemonic for remembering this pattern is **All Stores Take Cash**.
Example 3  Identify the quadrant (or quadrants) for any angle that satisfies $\tan \theta < 0$, $\cot \theta < 0$. (#30) $\tan \theta$ and $\cot \theta$ are negative in quadrants II and IV.

III. Ranges of Trigonometric Functions
For any angle $\theta$ for which the indicated functions exist:

A. $-1 \leq \sin \theta \leq 1$ and $-1 \leq \cos \theta \leq 1$;

B. $\tan \theta$ and $\cot \theta$ can equal any real number;

C. $\sec \theta \leq -1$ or $\sec \theta \geq 1$ and $\csc \theta \leq -1$ or $\csc \theta \geq 1$.

(Notice that $\sec \theta$ and $\csc \theta$ are never between $-1$ and 1.)

Example 4  Decide whether the following statement is possible or impossible for an angle $\theta$:

$\cos \theta = -1.001$ (#48)

Impossible. $\cos \theta$ is always be between $-1$ and 1, inclusive.

IV. Pythagorean Identities

$\sin^2 \theta + \cos^2 \theta = 1 \quad \tan^2 \theta + 1 = \sec^2 \theta \quad \cot^2 \theta + 1 = \csc^2 \theta$

Example 5  Find $\csc \theta$, if $\cot \theta = -\frac{1}{2}$, with $\theta$ in quadrant IV. (#57)

$\csc \theta$ is negative in quadrant IV, so $\csc \theta = -\sqrt{1 + \cot^2 \theta} = -\sqrt{1 + \left(-\frac{1}{2}\right)^2} = -\sqrt{1 + \frac{1}{4}} = -\sqrt{\frac{5}{4}} = -\frac{\sqrt{5}}{2}$

V. Quotient Identities

$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$

Example 6  Find all trigonometric function values for the angle $\theta$ if $\tan \theta = \sqrt{3}$, with $\theta$ in quadrant III. (#66)

Since $x$ and $y$ are both negative in quadrant III and since $\tan \theta = \frac{y}{x}$,

$\tan \theta = \sqrt{3}$ implies $x = -1$ and $y = -\sqrt{3}$ and $r = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$.

$\sin \theta = \frac{y}{r} = -\frac{\sqrt{3}}{2} \quad \cos \theta = \frac{x}{r} = -\frac{1}{2}$

$\csc \theta = \frac{r}{y} = \frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3} \quad \sec \theta = \frac{r}{x} = -\frac{2}{1} = -2 \quad \cot \theta = \frac{x}{y} = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$